Isochronous dynamical systems and the arrow of time

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Summary

Any (autonomous) dynamical system can be *extended* or *modified*, obtaining thereby a *new* (autonomous) dynamical system involving a constant *T*---the value of which can be freely assigned---and featuring the following two properties: (i) all solutions of the new model are *isochronous* (completely periodic in all their degrees of freedom with the assigned period *T*); (ii) starting from *generic* initial data, the time evolution of the new dynamical system over time intervals of order $\tilde{T} << T$ is essentially identical to that of the original dynamical system, up to a *constant* rescaling of time and of corrections of order $\tilde{T} / T$. These findings entail that, in some sense, “*isochronous systems are not rare*” and moreover that such systems may feature an “*extremely complicated* “ time-evolution. They are also valid in the context of *Hamiltonian* dynamics; they are in particular applicable to the most general many-body problem (provided it is, overall, translation-invariant), entailing remarkable observations about statistical mechanics, thermodynamics and the issue of the “arrow of time” for macroscopic physics. Since *completely periodic* systems are *maximally superintegrable* (possessing the maximal number of functionally independent constants of motion compatible with the time evolution not being frozen), these findings also entail that *any* (Hamiltonian) dynamics can be *embedded* into a *superintegrable* (Hamiltonian) dynamics; and again, that “*integrable (indeed, superintegrable) Hamiltonian systems are not rare*” and that such systems may feature an “*extremely complicated* “ time-evolution.

All these findings have been obtained together with **François Leyvraz**. Some of them are reported in a recent monograph (F. Calogero, *Isochronous systems*, Oxford University Press, 2008); others are more recent, see references listed below.

A trick to transform a dynamical system into an *isochronous* dynamical system

Let us start from an *autonomous*, but otherwise largely arbitrary, dynamical system, say

$$\underline{X}' = \underline{h}\left(\underline{X}\right); \;\; X'_n = h_n\left(\underline{X}\right), \;\; n = 1,\ldots,N, \;\; (*)$$

with the appended prime indicating differentiation with respect to the variable τ of $\underline{X}(\tau)$.

We then *change* it so that it reads as follows:

$$\dot{\underline{x}} = \dot{\tau}(t)\,\underline{h}\left(\underline{x}\right); \;\; \dot{x}_n = \dot{\tau}(t)\,h_n\left(\underline{x}\right), \;\; n = 1,\ldots,N, \;\; (**)$$

with the superimposed dot indicating differentiation with respect to the variable *t* of $\underline{x}(t)$ and $\tau(t)$.

It is then plain that the general solution of this system reads

$$\underline{x}(t) = \underline{X}\left[\tau(t)\right]; \;\; x_n(t) = X_n\left[\tau(t)\right], \;\; n = 1,\ldots,N,$$

where $\underline{X}(\tau)$ is the general solution of (*). Hence if the scalar function $\tau(t)$ is periodic with period *T*,

$$\tau(t+T) = \tau(t),$$

the general solution of (**) *inherits* from $\tau(t)$ the property to be *isochronous* with period *T*:

$$\underline{x}(t+T) = \underline{x}(t).$$

In conclusion, whenever $\underline{h}\left(\underline{X}\right)$ is such that the solution of the system (*) of *N* nonlinear ODEs exists globally, *all* solutions of the system (**) are periodic with period *T*: this system is *isochronous*.

However, the system (**) is *not* autonomous. To eliminate this "defect", we perform the second step of our treatment, replacing this system (**) with the system

$$\dot{\underline{x}} = \phi\, \underline{h}(\underline{x});\;\; \dot{x}_n = \phi\, h_n(\underline{x}),\;\; n = 1,\ldots,N,\;\; (***)$$

which is of course equivalent provided the time-evolution of the scalar quantity ϕ is such that

$$\phi(t) = \dot{\tau}(t).$$

There are now two options to obtain a quantity ϕ that qualifies for this purpose. One option---perhaps the most obvious---is to treat ϕ as an additional dependent variable, and to *extend* the system (*) by attaching to it a few additional ODEs involving ϕ and possibly other, additional dependent variables, so as to guarantee that the time evolution of ϕ has the desired property. Specific instances of how to achieve this goal are detailed in the last of the references listed below [F. Calogero and F. Leyvraz, "How to extend any dynamical system so that it becomes isochronous, asymptotically isochronous or multi-periodic", J. Nonlinear Math. Phys. (in press)], where it is also shown how a third step of our treatment---consisting essentially in a change of the additional dependent variables, entangling them with the original variables---allows to manufacture many, quite neat, extended dynamical systems having the required properties of *isochrony*. One example obtained in this manner is displayed below. A variant of this option, particularly suitable for *Hamiltonian* systems, will then be illustrated.

Another option is to identify a collective variable $\phi(\underline{x})$ that, as a consequence of the very evolution entailed by the dynamical system (***), has a time evolution, $\phi(t) \equiv \phi[\underline{x}(t)]$, such that, via the preceding formula, it defines a function $\tau(t)$ having the desired properties. This is the approach on which we will then focus, restricting moreover attention to a Hamiltonian system of major physical relevance.

Example: a modified Lorenz system

$$\dot{x}_1 = -\alpha\, y_1\, x_1 \left(x_1 - x_2\right),$$
$$\dot{x}_2 = y_1\, x_1 \left(\beta\, x_1 - x_2 - x_1\, x_3\right),$$
$$\dot{x}_3 = y_1\, x_1 \left(x_1\, x_2 - \gamma\, x_3\right),$$

$$\dot{y}_1 = \Omega\, y_2 + \alpha\left(x_1 - x_2\right) y_1^{\,2},$$
$$\dot{y}_2 = -\Omega\, y_1 + \alpha\left(x_1 - x_2\right) y_1\, y_2 .$$

For $\Omega = 0$ one can ignore y_2 (whose evolution does not influence the other variables) and set $y_1 = 1/x_1$: then the fourth of these 5 ODEs coincides with the first, and the first 3 become the 3 ODEs of the Lorenz model:

$$X_1' = -\alpha\ (X_1 - X_2),$$
$$X_2' = (\beta\ X_1 - X_2 - X_1\ X_3),$$
$$X_3' = (X_1\ X_2 - \gamma\ X_3).$$

Here the appended prime indicates differentiation with respect to the independent variable, which for convenience we call τ, so that $X \equiv X(\tau)$.

Indeed the solution of our *isochronous* model (with $\Omega > 0$) can be "explicitly" exhibited in terms of the solutions $X_n(\tau)$ of the Lorenz model (with the same initial data, $X_n(0) = x_n(0)$):

$$x_n(t) = X_n[\tau(t)], \quad n = 1,2,3,$$
$$y_1 = \dot{\tau}(t)/\, x_1(t), \ \ y_2 = \ddot{\tau}(t)/\, x_1(t),$$
$$\tau(t) = \{A \sin(\Omega t) + B[1 - \cos(\Omega t)]\}/\,\Omega .$$

A trick to *modify* a Hamiltonian into an *isochronous* Hamiltonian

$$\left[\ H\left(\underline{p},\underline{q}\right),\Theta\left(\underline{p},\underline{q}\right)\ \right]=1$$

Isochronous Hamiltonian:

$$\tilde{H}\left(\underline{p},\underline{q};\Omega\right)=\frac{1}{2}\left\{\left[H\left(\underline{p},\underline{q}\right)\right]^{2}+\Omega^{2}\left[\Theta\left(\underline{p},\underline{q}\right)\right]^{2}\right\};\quad T=\frac{2\pi}{\Omega}$$

"Isochronous Hamiltonian systems are not rare"

A remarkable example: the many-body problem

We write as follows the (simplest version of the) Hamiltonian characterizing the standard nonrelativistic *N*-body problem:

$$H(\underline{p},\underline{q}) = \frac{1}{2}\sum_{n=1}^{N}\left[p_n^2\right] + V(\underline{q}),\; V(\underline{q}+a) = V(\underline{q}).$$

Let us now review some standard related developments, trivial as they are.

We hereafter denote with P the total momentum, and with Q the (canonically-conjugate) centre-of-mass coordinate:

$$P = \sum_{n=1}^{N} p_n ,\; Q = \frac{1}{N}\sum_{n=1}^{N} q_n .$$

Thanks to the translation invariance property

$$[H,P] = 0$$

Here and hereafter the Poisson bracket $[F,G]$ of two functions $F(\underline{p},\underline{q})$ and $G(\underline{p},\underline{q})$ of the canonical variables is defined as follows:

$$[F,G] = \sum_{n=1}^{N}\left[\frac{\partial F(\underline{p},\underline{q})}{\partial p_n}\frac{\partial G(\underline{p},\underline{q})}{\partial q_n} - \frac{\partial G(\underline{p},\underline{q})}{\partial p_n}\frac{\partial F(\underline{p},\underline{q})}{\partial q_n}\right].$$

And let us recall that the evolution of any function $F(\underline{p},\underline{q})$ of the canonical coordinates is determined by the equation

$$F' = [H, F],$$

where the appended prime denotes differentiation with respect to the "timelike" variable corresponding to the evolution induced by the Hamiltonian *H*.

It is now convenient to introduce the "relative coordinates" x_n and the "relative momenta" y_n via the standard definitions

$$x_n = q_n - Q,\ \ y_n = p_n - \frac{P}{N}\ .$$

Note that these are not canonically conjugated quantities, since $[y_n, x_m] = \delta_{nm} - 1/N$, and they are not independent since obviously their sum vanishes:

$$\sum_{n=1}^{N} y_n = 0,\ \sum_{n=1}^{N} x_n = 0.$$

It is moreover convenient to introduce the "relative-motion" Hamiltonian $h(\underline{y},\underline{x})$ via the formula

$$h(\underline{y},\underline{x}) = \frac{1}{2}\sum_{n=1}^{N} y_n^2 + V(\underline{x}) = \frac{1}{4N}\sum_{n,m=1}^{N} (p_n - p_m)^2 + V(\underline{q})$$

so that

$$H(\underline{p},\underline{q}) = \frac{P^2}{2N} + h(\underline{y},\underline{x})\ .$$

Note that this definition of the relative-motion Hamiltonian $h(\underline{y},\underline{x})$ entails that it Poisson commutes with both P and Q:

$$[P,h] = 0,\ [Q,h] = 0.$$

For completeness and future reference let us also display the equations of motion implied by the original Hamiltonian $H(\underline{p},\underline{q})$:

$$q_n' = p_n\,,\;\; p_n' = -\partial V(\underline{q})/\partial q_n\,,\;\; q_n'' = -\partial V(\underline{q})/\partial q_n\,,$$

where (for reasons that will be clear below) we denote as τ the independent variable corresponding to this Hamiltonian flow and with appended primes the differentiations with respect to this variable:

$$q_n \equiv q_n(\tau),\;\; p_n \equiv p_n(\tau),\;\; q_n' \equiv \partial q_n(\tau)/\partial\tau\,,\;\; p_n' \equiv \partial p_n(\tau)/\partial\tau\,.$$

Hence

$$Q' = \frac{P}{N}\,,\;\; P' = 0$$

yielding

$$Q(\tau) = Q(0) + \frac{P(0)}{N}\tau\,,\;\; P(\tau) = P(0),$$

as well as

$$x_n' = y_n = \frac{\partial h(\underline{y},\underline{x})}{\partial y_n}\,,\;\; y_n' = -\frac{\partial V(\underline{x})}{\partial x_n} = -\frac{\partial h(\underline{y},\underline{x})}{\partial x_n}\,.$$

Note that these equations have the standard Hamiltonian form even though, as mentioned above, x_n and y_n are not canonically conjugated variables.

This ends the review of quite standard results for the classical nonrelativistic many-body problem. Let us also emphasize that, above and below, the restriction to unit-mass particles, and to one-dimensional space, is merely for simplicity: generalizations – also of the following results – to the more general case with *different* masses and *arbitrary* space dimensions is quite elementary, essentially trivial.

The isochronous Hamiltonian

The Ω-modified Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ is now defined by the formula

$$\tilde{H}(\underline{p},\underline{q};\Omega)=\frac{1}{2}\left\{\left[P+\frac{h(\underline{y},\underline{x})}{b}\right]^2+\Omega^2\,Q^2\right\},$$

where *b* is an arbitrary constant (introduced for dimensional reasons: it has the dimensions of a momentum, hence of the square-root of an energy) and Ω is a *positive* constant. Let us emphasize that hereafter the evolution of the various quantities is that caused by this new Ω-modified Hamiltonian (which does not quite reduce to the original Hamiltonian when Ω vanishes – more about this below); the corresponding independent variable is hereafter denoted as *t* (and interpreted as "time"), and differentiations with respect to this variable will be denoted, as usual, by superimposed dots, and of course, for any function $F\equiv F(\underline{p},\underline{q})$ of the canonical variable its time evolution will be determined by the standard equation

$$\dot{F}=\left[\tilde{H},F\right].$$

It is now easily seen that there hold the following Poisson commutation formulas:

$$\left[\tilde{H},Q\right]=P+\frac{h(\underline{y},\underline{x})}{b},\;\left[\tilde{H},P\right]=-\Omega^2\,Q,\;\left[\tilde{H},h\right]=0,$$

so that the quantities *Q*, *P* and *h* evolve as follows under the flow induced by the Ω-modified Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$:

$$\dot{Q}=P+\frac{h(\underline{y},\underline{x})}{b},\;\dot{P}=-\Omega^2\,Q,\;\dot{h}=0,$$

entailing

$$Q(t) = Q(0)\cos(\Omega t) + \dot{Q}(0)\frac{\sin(\Omega t)}{\Omega} = bC\,\frac{\sin[\Omega(t-t_0)]}{\Omega},$$

$$P(t) = P(0)\cos(\Omega t) + \dot{P}(0)\frac{\sin(\Omega t)}{\Omega} + \frac{h[\underline{y}(0),\underline{x}(0),]}{b}[\cos(\Omega t)-1],$$

$$h[\underline{y}(t),\underline{x}(t)] = h[\underline{y}(0),\underline{x}(0)].$$

It is moreover plain that the total momentum *P* and the centre-of-mass coordinate *Q* Poisson-commute with the relative-motion momenta and coordinates y_n and x_n hence as well with any function of these variables, hence the evolution equations of the relative-motion coordinates and momenta x_n and y_n under the flow induced by the Ω-modified Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ read

$$\dot{x}_n = \frac{1}{b}\left[P + \frac{h(\underline{y},\underline{x})}{b}\right]\frac{\partial h(\underline{y},\underline{x})}{\partial y_n} = \frac{\dot{Q}}{b}\,\frac{\partial h(\underline{y},\underline{x})}{\partial y_n},$$

$$\dot{y}_n = -\frac{1}{b}\left[P + \frac{h(\underline{y},\underline{x})}{b}\right]\frac{\partial h(\underline{y},\underline{x})}{\partial x_n} = -\frac{\dot{Q}}{b}\,\frac{\partial h(\underline{y},\underline{x})}{\partial x_n},$$

namely

$$\dot{x}_n = C\cos[\Omega(t-t_0)]\frac{\partial h(\underline{y},\underline{x})}{\partial y_n}\,,\ \ \dot{y}_n = -C\cos[\Omega(t-t_0)]\frac{\partial h(\underline{y},\underline{x})}{\partial x_n}\,,$$

$$C = \left(2\tilde{H}\right)^{1/2}/\,b,\ \ \sin(\Omega t_0) = -\Omega Q(0)\left(2\tilde{H}\right)^{-1/2}.$$

It is now crucial to observe -- by comparing these evolution equations caused by the Ω-modified Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ with the analogous ones caused by the original Hamiltonian $H(\underline{p},\underline{q})$ -- that it is justified to conclude that

$$\tilde{x}_n(t) = x_n(\tau), \;\; \tilde{y}_n(t) = y_n(\tau),$$

where (changing for convenience notation) we now denote as $\tilde{x}_n, \tilde{y}_n$ the canonical variables whose time evolution is determined by the Ω-modified Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ and as x_n, y_n the canonical variables whose time evolution is determined by the original, un-modified Hamiltonian $H(\underline{p},\underline{q})$. Here clearly (and most importantly)

$$\tau \equiv \tau(t) = C\,\frac{\sin(\Omega t)}{\Omega} = \frac{Q(t) - Q(0)}{b} \; ,$$

where the constant *C* is given by simple explicit formula in terms of the initial values of the centre-of-mass of the system and of the Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ (which is of course a constant of motion). The crucial observation is that $\tau(t)$ is a (real, nonsingular) periodic function of *t* with period

$$T = 2\pi/\Omega \; .$$

In this manner the dynamics of the canonical coordinates and momenta evolving according to our Ω-modified Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ are finally obtained, and due to the periodic behaviour under this flow of the collective coordinates *Q* and *P*, as well as the remarkable relation we just found among the time evolution of the "relative motion" variables of the Ω-modified Hamiltonian and those of the original unmodified Hamiltonian, it is now plain that the dynamics yielded by the Ω-modified Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ is *isochronous* with period *T* (for arbitrary initial data): indeed the time evolution due to the original Hamiltonian is generally, for arbitrary initial data, uniquely well-defined for all real time -- unless it runs into singularities, which should not be the case for physically sound models, and in any case should only happen exceptionally.

Behaviour of the *isochronous* system over time scales much shorter than *T*

The relevant formula is of course always

$$\tilde{x}_n(t) = x_n(\tau),\ \ \tilde{y}_n(t) = y_n(\tau),$$

with

$$\tau \equiv \tau(t) = C\,\frac{\sin(\Omega t)}{\Omega} = \frac{Q(t) - Q(0)}{b}\ .$$

These formulas confirm the assertion that the dynamics yielded by the Hamiltonian $\tilde{H}(\underline{p},\underline{q};0)$ differs only marginally from that yielded by the original Hamiltonian $H(\underline{p},\underline{q})$. Indeed clearly $\tau = \tau(t)$ on a sufficiently short time scale varies linearly in *t,* since in the neighbourhood of any time $\bar{t}$ -- except when $\dot{Q}(\bar{t}) = C\cos(\Omega\bar{t})$ vanishes –

$$\tau(t) = \overline{C} + t\,C\cos(\Omega\bar{t}) + O\left[\left(\frac{t-\bar{t}}{T}\right)\right],$$

$$\overline{C} = C\left[\sin(\Omega\bar{t}) - \Omega\bar{t}\cos(\Omega\bar{t})\right]/\,\Omega\ .$$

Transient chaos

One therefore finds that – essentially throughout the time evolution -- the Ω-modified dynamics differs from the unmodified one solely by a time rescaling -- by a possibly negative coefficient -- and by a time shift. The coefficient and the shift are time-independent over a time scale much smaller than the *isochrony* period *T*=2π/Ω, but vary periodically with period *T*. A peculiar state of affairs arises, however, whenever $\dot{Q}(\bar{t})$ vanishes, namely when $d\tau / dt$ changes its sign: this of course happens twice within every time period *T*, this being in fact a consequence of the periodicity of $\tau(t)$, which itself is the cause of the *isochrony*. These aspects are apparent in the two figures displayed in the examples reported below.

It is interesting to speculate on the application of this Ω-modification technique to any Hamiltonian describing a “realistic” translation-invariant many-body problem featuring, in its centre-of-mass system, “chaotic” motions with a natural time scale T_C. Then -- provided the constant Ω is assigned so that the *isochrony* period $T = 2\pi / \Omega$ is much larger than this time scale, $T >> T_C$ -- the Ω-modified problem shall exhibit some kind of *chaotic* behavior for quite some time before the *isochronous* character of all its motions takes over, causing thereafter a recurrent evolution. This phenomenology -- qualitative rather than quantitative as it necessarily is, since a precise definition of *chaos* requires generally that a system displaying it be observed for *infinite* time -- is nevertheless remarkable.

A simple example

I now display, with minimal comments, the findings obtained by applying our Ω-modification technique to the very simple Hamiltonian describing a couple of equal mass one-dimensional particles interacting pairwise with a force proportional to their mutual distance; when this force is attractive this model corresponds of course, in the centre-of-mass system, to the standard "harmonic" oscillator model. (We write "harmonic" under inverted commas to emphasize that the *isochronous* Ω-modified Hamiltonians yielded by our technique *all* yield motions deserving to be called harmonic, inasmuch as they are characterized by just a single frequency of oscillation: in the case of many-body problems the term "nonlinear harmonic oscillators", introduced together with Inozemtsev, is perhaps the most appropriate to describe the corresponding dynamics...).

The original Hamiltonian:

$$H(p_1, p_2; q_1, q_2) = \frac{1}{2}\left(p_1^{\,2} + p_2^{\,2}\right) + \frac{\omega^2}{4}(q_1 - q_2)^2 \ .$$

Some standard definitions and related formulas:

$$x_1 = q_1 - Q = \frac{q_1 - q_2}{2}, \ \ x_2 = q_2 - Q = \frac{q_2 - q_1}{2}, \ x_1 + x_2 = 0 \ ,$$

$$y_1 = p_1 - \frac{P}{2} = \frac{p_1 - p_2}{2}, \ \ y_2 = p_2 - \frac{P}{2} = \frac{p_2 - p_1}{2}, \ y_1 + y_2 = 0 \ ,$$

$$h(x_1, x_2; y_1, y_2) = \frac{1}{2}\left(y_1^{\,2} + y_2^{\,2}\right) + \frac{\omega^2}{4}(x_1 - x_2)^2 \ ,$$

$$H(p_1, p_2; q_1, q_2) = \frac{1}{4}P^2 + h(y_1 y_2; x_1 x_2) \ ;$$

$$x_n(t) = x_n(0)\cos(\omega t) + y_n(0)\frac{\sin(\omega t)}{\omega} \ , \ \ y_n(t) = y_n(0)\cos(\omega t) - \omega\, x_n(0)\frac{\sin(\omega t)}{\omega} \ , n = 1{,}2 \ .$$

The Ω-modified Hamiltonian

$$\tilde{H}(p_1 p_2; q_1 q_2; \Omega) = \frac{1}{2}\left\{\left[P + \frac{h(y_1 y_2; x_1, x_2)}{b}\right]^2 + \Omega^2\left(\frac{q_1+q_2}{2}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left[p_1 + p_2 + \frac{(p_1-p_2)^2}{4b}\right]^2 + \frac{\omega^2}{2b}\left[p_1 + p_2 + \frac{(p_1-p_2)^2}{4b}\right](q_1-q_2)^2 + \left(\frac{\omega^2}{4b}\right)^2 (q_1-q_2)^4 + \Omega^2\left(\frac{q_1+q_2}{2}\right)^2\right\}$$

The *isochronous* motions yielded by this Ω-modified Hamiltonian:

$$x_n(t) = x_n(0)\cos\left\{\omega C\frac{\sin[\Omega(t-t_0)]+\sin(\Omega t_0)}{\Omega}\right\} + \frac{y_n(0)}{\omega}\sin\left\{\omega C\frac{\sin[\Omega(t-t_0)]+\sin(\Omega t_0)}{\Omega}\right\},\ n=1,2,$$

$$y_n(t) = y_n(0)\cos\left\{\omega C\frac{\sin[\Omega(t-t_0)]+\sin(\Omega t_0)}{\Omega}\right\} - \omega x_n(0)\sin\left\{\omega C\frac{\sin[\Omega(t-t_0)]+\sin(\Omega t_0)}{\Omega}\right\},\ n=1,2,$$

with C and t_0 defined in terms of the initial data.

The *isochronous* character of this motion, with period $T=2\pi/\Omega$, is evident; and note that this outcome obtains even if ω is purely imaginary, $\omega=i\alpha$ with α real, in which case the original Hamiltonian H and the Ω-modified Hamiltonian $\tilde{H}$, as well of course as the corresponding solutions, are nevertheless *all real*.

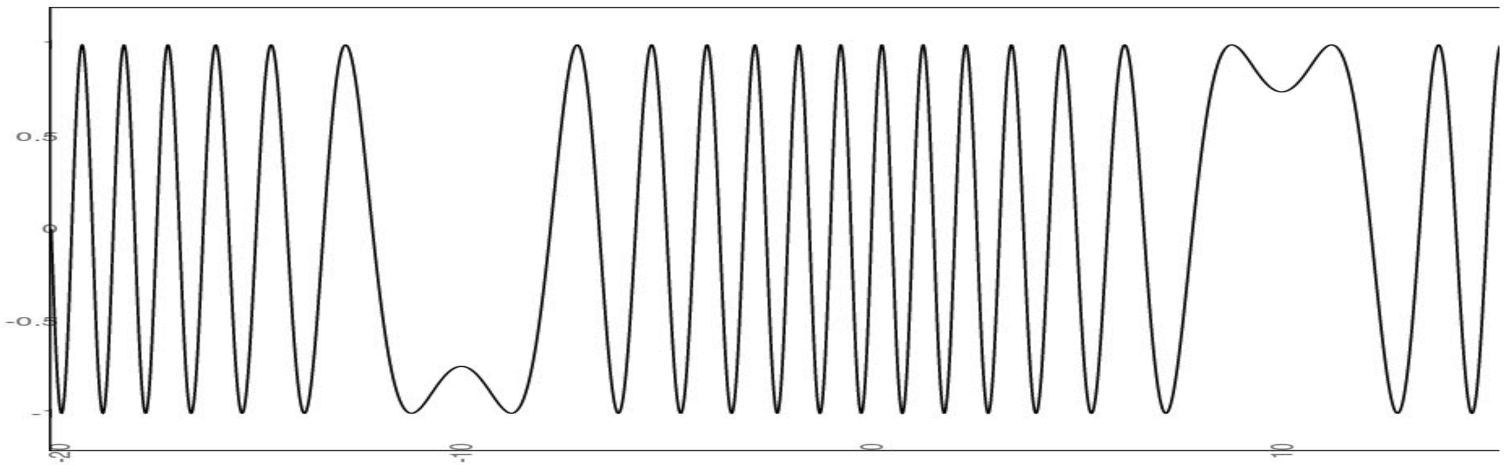

Figure 1: Graph (over almost one time period) of $x_n(t)$ for
$x_n(0)=0,\ y_n(0)=\omega=40\Omega=2\pi,\ \Omega=\pi/20,\ C=1,\ t_0=0.$

Note the overall periodicity with period $T=2\pi/\Omega=40$, the large regions where the behavior is nearly periodic with the original period $2\pi/\omega=1$ of the solutions of the unmodified Hamiltonian, and the transition regions around the times *t*=10 and *t*=30 when $\dot{\tau}(t)$.

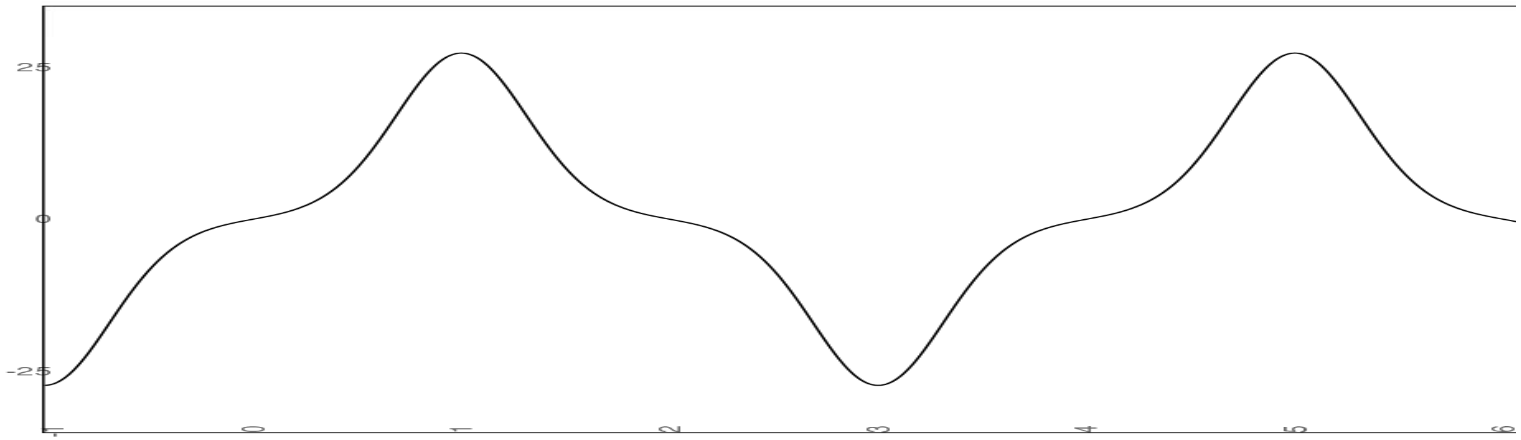

Figure 2: Graph (over almost two time periods) of $x_n(t)$ for
$x_n(0)=0,\ \ y_n(0)=\omega=4i\Omega=2\pi i,\ \ \Omega=\pi/2,\ \ C=1,\ \ t_0=0.$
Note the overall periodicity with period $T=2\pi/\Omega=4$, the regions where the time evolution resembles the original $\sin(\omega t)/i=\sinh(2\pi t)$ behavior of the corresponding solution of the unmodified Hamiltonian, and the transition regions around the times *t*=-1,1,3,5,7 when $\dot{\tau}(t)$ vanishes.

The quantum case

Finally we tersely show that, in a quantal context, our Ω -modified Hamiltonian

$$\tilde{H}\left(\underline{p},\underline{q};\Omega\right)=\left\{\left[P+h\left(\underline{y},\underline{x}\right)/b\right]^2+\Omega^2 Q^2\right\}/2,$$

features an (infinitely degenerate) equispaced spectrum with spacing $\hbar\Omega$.

This spectrum consists of the eigenvalues E_k of the stationary Schrödinger equation

$$\frac{1}{2}\left\{\left[-i\hbar\frac{\partial}{\partial Z}+\frac{\lambda}{b}\right]^2+\Omega^2 Z^2\right\}\Psi_k(Z;\lambda)\psi_\lambda(\underline{z};\lambda)=E_k\Psi_k(Z;\lambda)\psi_\lambda(\underline{z};\lambda),$$

obtained from this Hamiltonian via the standard quantization rule

$$P\Rightarrow -i\hbar\partial/\partial Z,\;\; Q\Rightarrow Z\;;\;\; \underline{p}\Rightarrow -i\hbar\partial/\partial\underline{z},\;\; \underline{q}\Rightarrow\underline{z}\quad ,$$

and by identifying λ as an eigenvalue of the quantized version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$. Indeed this Schrödinger equation is obtained by assuming that the eigenfunctions of the quantized version of the Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$ factor into the product of an eigenfunction, $\Psi_k(Z;\lambda)$, depending on the variable *Z* and on which acts the differential operator *∂/∂Z* , and of the eigenfunction $\psi_\lambda(\underline{z};\lambda)$,corresponding to the eigenvalue λ of the quantized version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$. The justification for this factorization is in the commutativity of the operators representing the quantal versions of the canonical variables *P* and *Q* , see above, with the operator representing the quantal version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$ -- a commutativity reflecting the Poisson-commutativity of the corresponding quantities in the classical context.

It is now plain that the above Schrödinger equation features the spectrum and eigenfunctions

$$E_k=\hbar\Omega\left(k+1/2\right),\;\; k=0,1,2,...,$$

$$\psi_k(Z;\lambda)=\exp\left(\frac{i\lambda z}{b\sqrt{\hbar\Omega}}-\frac{z^2}{2}\right)H_k(z),\;\; z=Z\sqrt{\frac{\Omega}{\hbar}}\quad ,$$

where $H_k(z)$ denotes the standard Hermite polynomial of order *k*.

This spectrum is of course equispaced with spacing $\hbar\Omega$, and it is infinitely degenerate inasmuch as it does not feature any dependence on the eigenvalues λ.

How to *embed* an arbitrary Hamiltonian dynamics in a *superintegrable* Hamiltonian dynamics

Lemma. Consider the following *3* Hamiltonians: *H(P,Q)*, depending on the *2* canonical variables *P* and *Q*; $h(\underline{p},\underline{q})$, depending on the *2N* canonical variables p_n and q_n; and

$$\tilde{H}\left(\underline{\tilde{p}},\tilde{P};\underline{\tilde{q}},\tilde{Q}\right)=H\left(\tilde{P}+h\left(\underline{\tilde{p}};\underline{\tilde{q}}\right);\tilde{Q}\right),$$

having the *N+1* momenta $\tilde{p}_n,\tilde{P}$ and the corresponding *N+1* coordinates $\tilde{q}_n,\tilde{Q}$ as its canonical variables. Here and hereafter the superimposed tilde is a reminder that the time evolution of the corresponding quantities is determined by this "tilded" Hamiltonian. Note that this definition of the tilded Hamiltonian implies that

$$c=h\left(\underline{\tilde{p}};\underline{\tilde{q}}\right)$$

is a constant of motion. Then

$$\underline{\tilde{p}}(t)=\underline{p}\left(\tilde{Q}(t)\right),\ \underline{\tilde{q}}(t)=\underline{q}\left(\tilde{Q}(t)\right),$$

where the time evolution of $\underline{p}$ and $\underline{q}$ is determined by the Hamiltonian $h(\underline{p},\underline{q})$ (of course provided the initial data also fit).

Proof. The *2N+2* equations of motion yielded by the tilded Hamiltonian read as follows:

$$\dot{\tilde{P}} = -\partial H\left(\tilde{P}+c;\tilde{Q}\right)/\partial\tilde{Q},\ \ \dot{\tilde{Q}} = \partial H\left(\tilde{P}+c;\tilde{Q}\right)/\partial\tilde{P}\ ,$$

$$\dot{\tilde{p}}_n = -\dot{\tilde{Q}}\,\partial h\left(\underline{\tilde{p}},\underline{\tilde{q}}\right)/\,\partial\tilde{q}_n\ ,\ \ \dot{\tilde{q}}_n = \dot{\tilde{Q}}\,\partial h\left(\underline{\tilde{p}},\underline{\tilde{q}}\right)/\,\partial\tilde{p}_n\ ,$$

where, to get the equations of motion for the *2N* lower-case variables, we used the second equation of motion of the *2* upper-case variables. Hence they clearly entail that $\underline{\tilde{p}}(t)$ satisfies the same evolution equation as $\underline{p}\left(\tilde{Q}(t)\right)$, and likewise $\underline{\tilde{q}}(t)$ satisfies the same evolution equation as $\underline{q}\left(\tilde{Q}(t)\right)$. Q.E.D.

Let us re-emphasize that, for notational convenience, superimposed tildes identify the dynamical variables whose time-evolution is determined by the tilded Hamiltonian, while the dynamical variables $\underline{p}(t),\underline{q}(t)$ without superimposed tildes are those whose time-evolution is determined by the Hamiltonian $h(\underline{p};\underline{q})$:

$$\dot{p}_n = -\partial h\left(\underline{p};\underline{q}\right)/\,\partial q_n\ ,\ \ \dot{q}_n = \partial h\left(\underline{p};\underline{q}\right)/\,\partial p_n\ .$$

Theorem. Let the Hamiltonian *H(P,Q)*, depending on the 2 canonical variables *P* and *Q*, have the following two properties: (i) for *P(0)=0*, this Hamiltonian yields the simple solution

$$P(t)=0,\ \ Q(t)=Q(0)+t\ ,$$

entailing that the relation

$$P=0$$

identifies an invariant manifold of this Hamiltonian; (ii) for *P(0)≠0*, the evolution of the canonical variables is instead *periodic*, namely there exists a nonnegative period *T*---depending on the initial data *P(0)* and *Q(0)*---such that, for all time *t*,

$$P(t+T)=P(t),\ \ Q(t+T)=Q(t)\,.$$

Note that such Hamiltonians certainly exist; an example is exhibited below.

Then the relation

$$\tilde{P} + h\left(\underline{\tilde{p}};\underline{\tilde{q}}\right) = 0, \qquad (*)$$

identifies an invariant manifold for the tilded Hamiltonian, and for *any* initial data *off* this invariant manifold (*), namely such that

$$\tilde{P}(0) + h\left(\underline{\tilde{p}}(0);\underline{\tilde{q}}(0)\right) \neq 0,$$

the time evolution yielded by the tilded Hamiltonian is *completely periodic,*

$$\tilde{P}(t+T) = \tilde{P}(t),\ \ \tilde{Q}(t+T) = \tilde{Q}(t),$$

$$\underline{\tilde{p}}(t+T) = \underline{\tilde{p}}(t),\ \ \underline{\tilde{q}}(t+T) = \underline{\tilde{q}}(t),$$

hence *all* its orbits are *closed*, implying that the tilded Hamiltonian is *superintegrable*;

while for initial data *on* the invariant manifold (*)---i. e., for initial data with

$$\tilde{P}(0) = -h\left(\underline{\tilde{p}}(0);\underline{\tilde{q}}(0)\right)$$

---the time evolution of the *2N* variables $\tilde{p}_n, \tilde{q}_n$ is exactly the same as that yielded by the (*arbitrary*!) Hamiltonian *h*, i. e. by the equations of motion

$$\dot{\tilde{p}}_n = -\partial h\left(\underline{\tilde{p}};\underline{\tilde{q}}\right)/\partial\tilde{q}_n, \quad \dot{\tilde{q}}_n = \partial h\left(\underline{\tilde{p}};\underline{\tilde{q}}\right)/\partial\tilde{p}_n .$$

This **Theorem**---whose validity is now plain---details the precise meaning of the term "embed" used above.

Remark. As stated above there are many Hamiltonians *H(P,Q)* yielding evolutions that satisfy the requirements specified in the formulation of the **Theorem**. Clearly the requirement that *P=0* be an invariant manifold characterized by the simple evolution

$$P(t)=0,\ \ Q(t)=Q(0)+t\ ,$$

is satisfied by any Hamiltonian *H(P,Q)* such that, for *P=0*,

$$\partial H(P,Q)/\partial P=1,\ \ \partial H(P,Q)/\partial Q=0,$$

while the requirement that, for *P≠0*, *all* the trajectories yielded by this Hamiltonian be *closed* can be enforced by making sure that *all* constant-energy contours of *H(P,Q)* are bounded.

A specific Hamiltonian *H(P,Q)* having these properties reads as follows:

$$H(P,Q)=P-P^2\left(Q^2+1\right),$$

with the restriction $0\le H(P,Q)\le 1/4$.

Indeed it is then a simple exercise to obtain the general solution yielded by this Hamiltonian:

$$Q(t)=\cot\theta\,\sin\left[\left(t+t_0\right)\sin\theta\right],$$

$$P(t)=\frac{1-\cos\theta\cos\left[\left(t+t_0\right)\sin\theta\right]}{2\left\{1+\cot^2\theta\sin^2\left[\left(t+t_0\right)\sin\theta\right]\right\}},$$

where we denoted, for notational convenience, the constant value of the Hamiltonian as follows:

$$4\,H\left(P,Q\right)=\sin^2\theta\ .$$

These formulae contain two constants, θ and t_0, hence they provide the general solution of the equations of motion yielded by the Hamiltonian $H(P,Q)=P-P^2(Q^2+1)$. In the initial-value problem, the constant angle θ (including the quadrant it belongs to) and the constant t_0 (up to an irrelevant mod($2\pi/\sin\theta$) ambiguity) are determined in terms of the initial data $Q(0)$ and $P(0)$ by the requirement that these formulae hold at $t=0$.

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