Fourier Transform, Riemann Surfaces and Indefinite Metric

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What is Fourier Transform in Riemann Surfaces? Which Problems need it?

Discrete Analog of The Fourier/Laurent bases in Riemann Surfaces was constructed by Krichever-Novikov (KN, 1986-1990) for The Operator Quantization of The Closed String. The Ideas were borrowed from The Theory of Finite-Gap Potentials in The Theory of Solitons where Riemann Surfaces appear as Spectral Curves.

Continuous Fourier Transform on Riemann Surfaces was constructed in our work (GN, 2003). As we found recently (GN, 2008-2009), this transform preserves an Indefinite Inner Product for genus $g > 0$, for the cases where Fourier Transform has good Multiplicative Properties. The Operators are Singular here.
The ordinary Fourier Transform: Basic functions has 2 fundamental properties a) and b):

\[ \Psi_n(k) = k^n, \quad x = n \in \mathbb{Z}, \quad |k| = const > 0 \text{ (discrete)} \]
\[ \Psi(x, k) = \exp(ikx), \quad x \in \mathbb{R} \text{ (continuous)} \]

a) They form an orthonormal basis
b) They have graded multiplicative law:

\[ \Psi_n(k)\Psi_m(k) = \Psi_{m+n}(k), \quad \Psi(x, k)\Psi(y, k) = \Psi(x+y, k) \]

Here Riemann Suface \( \Gamma = S^2 \) has genus \( g = 0, \quad ik = z^{-1} \). It belongs to the selected ”Canonical Contour” \( \kappa_c \) on Riemann Surface \( |z| = \exp c \) in the discrete case, \( Imk = c \) in the continuous case. Especially interesting is the Special Canonical Contour \( \kappa_0 \).

Only this contour will be considered for the Continuous Fourier Transform
Discrete Case: Fourier/Laurent (KN) bases on Riemann surfaces.


The String Diagram $(\Gamma, P_+, P_-, k_+, k_-)$:

Here $1/k_+$, $1/k_-$ are local parameters near "The Infinite Points" $P_-$ ("in") and $P_+$ ("out") respectively, $d\rho$ is defined as meromorphic differential with 2 simple poles at $P_+, P_-^\ast$

\[ d\rho = dk_+/k_+ + O(1), \]
\[ d\rho = -dk_-/k_- + O(1) \]

$\text{Re} \oint_C d\rho = 0$ for closed contours.

The "time" is $\tau = \text{Re} \rho$

$\tau(P_+) = +\infty$, $\tau(P_-) = -\infty$

An analogue of discrete Fourier bases is defined for functions (tensor fields) at the contour $\kappa_c : -\infty < \tau = c < +\infty$
An analogue of Laurent basis for the holomorphic functions is defined for the domains between the contours $\kappa_{c'}$ and $\kappa_{c''}$ where $c' < c''$. All constructions are extended to the tensor fields with any tensor weights. The tensors with weights equal to $0,1,-1,2,1/2$ are especially important for the string theory. Krichever and Novikov introduced these bases to construct operator quantization of bosonic string. For this problem it was critical to have bases with good multiplicative properties. These bases are defined by the following asymptotics at the points $P_+, P_-$:

$$
\Psi_j(\lambda) = \begin{cases} 
k_{c_j}^{j+g/2} (c_j^+ + o(1)) & \lambda \to P_+ \\
k_{c_j}^{-j+g/2} (c_j^- + o(1)) & \lambda \to P_-
\end{cases}
$$

(We present here the case of scalar functions only, $j \in \mathbb{Z}$ for $g = 2s$ and $j \in \mathbb{Z} + 1/2$ for $g = 2s + 1$ and $j$ big enough.)
The multiplication rule is:

\[ \psi_l(\lambda) \psi_m(\lambda) = \sum_{n=l+m-N}^{n=l+m+N} C_{lm}^n \psi_n(\lambda) \]

where \( N = N(g) \), does not depend on \( l, m \), \( C_{lm}^n \) do not depend on \( \lambda \).

Such multiplications are called almost graded.

Let \( z = 1/k \) be local parameter near \( P \), define \( dp \) as a meromorphic differential with a second-order pole at \( P \)

\[ dp = dk + O(1), \]

\( \text{Im} \oint_C dp = 0 \) for any closed contour \( C \).

\( \tau = \text{Im} p \) is well-defined.

The Special Canonical Contour is \( \kappa_0 : \tau = \text{Im} p = 0 \).
The standard finite-gap inverse spectral data:

1) A compact Riemann surface $\Gamma$ of genus $g$ with an "infinite" point $P$.
2) A local parameter $z = 1/k$ near $P$.
3) A collection of points $\gamma_1, \ldots, \gamma_g$ (the poles of $\psi$-function), $D = \gamma_1 + \ldots + \gamma_g$.

For the important cases this surface is hyperelliptic (2-sheeted over $E$-plane).
The eigenfunction $\Psi(\lambda, x)$, $\lambda \in \Gamma$, $x \in \mathbb{R}$.

1. $\Psi(\lambda, x)$ is meromorphic in $\Gamma \setminus P$ with simple poles $\gamma_1, \ldots, \gamma_g$.
2. $\Psi(\lambda, x) = (1 + o(1)) \exp(ikx)$, $\lambda \to \infty$.

Let $g = 0$ and $\Gamma = C \cup \infty$, $P = \infty$. Here $k$ is the standard coordinate. Then $p = k$, $\Psi(\lambda, x) = \exp(ikx)$ is the standard Fourier basis on the real line $\text{Im} \ k = 0$.

A continuous analog of the Fourier bases (Grinevich-Novikov, 2003).

Let $\gamma_1 = \ldots = \gamma_g = P$. Then the $\psi$-functions form an almost-
graded algebra:

$$\Psi(\lambda, x)\Psi(\lambda, y) = \sum_{j=0}^{g} c_j(x, y) \partial^j_z \Psi(\lambda, z) \bigg|_{z=x+y}$$

We study functions of $\lambda$, and $x$ is a parameter numerating our basic functions.

The functions $\Psi(\lambda, x)$ are **singular** in $x$. They have a pole at $x = 0$. For example, the classical periodic Lame’ operator $-\partial_x^2 + n(n + 1)\wp(x)$ is exactly a special case here for all $g > 0$.

Physical Soliton Theory (KdV) deals with regular operators $-\partial_x^2 + n(n + 1)\wp(x + i\omega')$ where $2\omega'$ is an imaginary period.

Our aim is to answer the following question:
Do such singular operators have a reasonable spectral theory on the whole real line $x$?

Classical people since Hermit considered their spectrum only at the interval $[0, T = 2\omega]$ with zero boundary conditions. We need the whole line for the Fourier Transform with good multiplicative properties.

The Baker-Akhiezer $\psi$-functions for regular real periodic operators never form an almost-graded multiplicative system for $g > 0$. We need singular operators for Good Fourier Transform.

Consider real finite-gap periodic regular or singular operator

$$L = -\partial_x^2 + u(x).$$

$\Gamma$ is real and defined by: $\mu^2 = (E - E_1) \cdots (E - E_{2g+1})$
Permutation of sheets is defined $\sigma(E, \mu) = (E, -\mu)$, $\sigma^2 = \text{id}$

The regular case corresponds to the following data:
1) All $E_j$ are real. Let $E_1 < E_2 < \ldots < E_{2g+1}$
2) Each closed interval $[E_{2j}, E_{2j+1}]$, $j = 1, \ldots, g$ contains exactly one pole: $\lambda_j \in [E_{2j}, E_{2j+1}]$, here $\lambda_j$ denotes the projection of $\gamma_j$ to the $E$-plane.

The real singular case corresponds to the following data:
1) $\Gamma$ is real i.e. collection of branching points is invariant under complex conjugation. Let $\tau(E, \mu) = (\bar{E}, \bar{\mu})$, $\tau^2 = \text{id}$.
2) Collection of poles is also ”real”: the whole collection is invariant under $\tau$. 
Basic Example: Let \( g = 1 \) (\( \Gamma \) is a torus):

1) All \( E_j \) are real, \( j = 1, 2, 3 \):

The lattice of periods of the Weierstrass \( \wp \) -function is rectangular with periods \( 2\omega, 2i\omega' \).

The gaps are \([-\infty, E_1]\) and \([E_2, E_3]\)
The contour $\kappa_0$ has 2 components here: infinite and finite. There is only one pole $\gamma$: For regular case it belongs to the finite gap, for the singular case it belongs to the infinite gap (The Shifted Hermit -Lame Operators).

In both cases the spectrum on the whole line is a union of 2 real sets $[E_1, E_2] \cup [E_3, \infty]$ (projection of $\kappa_0$) but eigenfunctions and functional spaces on the $x$-line are drastically different.
2) Let $E_1$ is real, $E_3 = \overline{E_2}$:

The lattice of periods is **rombic**.

\[ \omega \]
\[ 0 \]
\[ \omega + \omega \]

$\kappa_0$ given by fine lines.
The spectrum on the whole line coincides with the projection of the contour $\kappa_0$ on the $E$ line. It contains complex (nonreal) arc joining $E_2, E_3 = \bar{E}_2$. Spectral meaning of singular operators on the whole line was not discussed before.

Direct and Inverse Spectral Transform
We invent following ”measure” for $\lambda = (E, \pm) \in \Gamma$

\[\Psi^*(\lambda, x) = \Psi(\sigma \lambda, x); \gamma_j = (\lambda_j, \mu_j) = \text{poles,}\]

\[d\mu = \frac{(E - \lambda_1) \ldots (E - \lambda_g)dE}{2\sqrt{(E - E_1) \ldots (E - E_{2g+1})}},\]

For every smooth function $\phi(\lambda), \lambda \in \kappa_0$, with decay sufficiently
fast at \( \lambda \to P \), we define **A Spectral Transform:**

\[
\tilde{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\kappa_0} \phi(\lambda)\Psi^*(\lambda, x) d\mu(\lambda)
\] (1)

and **Inverse Spectral Transform:**

\[
\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\phi}(\lambda)\Psi(\lambda, x) dx
\] (2)

We call it Fourier Transform for the case \( \lambda_1 = \ldots = \lambda_g = \infty \); \( d\mu^{\text{Fourier}} = dE/2 \sqrt{(E - E_1) \ldots (E - E_{2g+1})} \). In this case our basis has good multiplicative properties.

In the regular case Spectral Transform is an isometry between the spaces \( L^2(\kappa_0) \) and \( L^2(\mathbb{R}) \) with inner
-products

\[<\psi_1, \psi_2>_{\kappa_0} = \int_{\kappa_0} \psi_1(\lambda)\overline{\psi_2}(\lambda) d\mu(\lambda)\]

and

\[<f_1, f_2>_{\mathbb{R}} = \int_{\mathbb{R}} f_1(x)\overline{f_2}(x) dx.\]

(The “measure” \(d\mu\) is positive in \(\kappa_0\)).

The Singular Potentials:

1) Formula for the Spectral Transform remains valid; For the Inverse Transform it remains valid after a natural regularization.
2) Spectral Transform is an isometry between the spaces with indefinite metric described below.
All singularities have a form

\[ u(x) = n(n + 1)/(x - x_j)^2 + O(1). \]  \hspace{1cm} (3)

The function \( \Psi(\lambda, x) \) is meromorphic in \( x \). For \( \lambda', \lambda'' \in \Gamma \) all residues of the product \( \Psi(\lambda', x)\Psi(\lambda'', x) \) are equal to 0.

The Scattering Data for potentials with singularities of the type \( 2/x^2 \) were constructed in:


The Indefinite Inner Product was not discussed.

All residues in the formula for the Inverse Spectral Transform are equal to 0. Our Regularization: If we
meet a singularity under the integral, we go around it in the complex domain. We can go above or below, but the result is the same.

The Inner Product on the Riemann Surface (in the space of functions in $\kappa_0$)

$$<\psi_1, \psi_2>_{\kappa_0} = \int_{\kappa_0} \psi_1(\lambda)\overline{\psi_2}(\tau \lambda) d\mu$$

1) All branching points are real: $\tau$ acts identically on $\kappa_0$, the form $d\mu$ is negative somewhere. For Fourier Transform the measure $d\mu_{\text{Fourier}}$ is negative in $[(g + 1)/2]$ finite components of the contour $\kappa_0$. 
2) Some pair of branching points is complex adjoint: the inner product is nonlocal and therefore indefinite.

\[ \langle f_1, f_2 \rangle_{\mathbb{R}} = \int_{\mathbb{R}} f_1(x) \overline{f_2(x)} \, dx \]

The Inner Product on the space of functions in \( \mathbb{R} \)

These functions belong to the image of the Spectral Transform. For the Generic case all singularities have
the form \( u(x) \sim 2/(x - x_j)^2 \). Locally the functions \( f_1, f_2 \) have the form:

\[
f(x) = d_{-1}/(x - x_j) + d_1(x - x_1) + \ldots
\]

We write \( \bar{x} \) instead of \( x \) to make both terms holomorphic. Residues in all singularities are equal to 0 for the products under integral, therefore we go around it in the complex domain either above or below. This scalar product is indefinite.
Pontryagin-Sobolev (PS) spaces: Every function $f(x)$ can be written for real $x$ as

$$f(x) = \int_{0}^{2\pi/T} \hat{f}(p, x) dp,$$

where $f(p, x + T) = \exp(ipT)f(p, x)$. Therefore $L^2(\mathbb{R})$ is represented as a direct integral of Bloch-Floquet spaces $B_\kappa$:

$$f(x) \in B_\kappa \text{ if } f(x + T) = \kappa f(x), \quad |\kappa| = 1.$$

Our inner product has finite number $r$ of negative squares in $B_\kappa$, so it is PS. For Fourier case $r = \lceil (g + 1)/2 \rceil$. 
This result allows to estimate the number $r'$ of singularities for $u(x)$ at the real line.

Example: KdV deformations. Take:

$$u(x, 0) = g(g+1)\varphi(x), u(x, 0) = g(g+1)/x^2, g \in \mathbb{Z}$$

Apply KdV dynamics to this potentials. We obtain

$$u(x, t) = \sum_{j=1}^{g(g+1)/2} 2\varphi(x - x_j(t))$$

Question: Calculate the number $r'$ of real poles $x_j(t)$. Result remains the same taking $g(g + 1)/x^2$ instead of $g(g + 1)\varphi(x)$.
Our Argument:
For $t = 0$ the number $r$ of negative squares of our inner product is equal to $\left\lfloor (g + 1)/2 \right\rfloor$. The number of negative squares is stable, therefore we have at least $r' \geq \left\lfloor (g + 1)/2 \right\rfloor$ real poles. We checked in many cases numerically that $r' = r$ for all $g$.


Final remark: Singular Bloch-Floquet eigenfunctions are known also for the $k + 1$-particle Calogero-
Moser operator with Weierstrass elliptic pairwise potential if coupling constant is equal to \( n(n+1) \). They form a \( k \)-dimensional complex algebraic variety. The Hermit-type result is not obtained here yet for \( k > 1 \): no one function was constructed until now serving the discrete spectrum in the bounded domain inside of the poles. Our case corresponds to \( k = 1 \). We believe that for all \( k > 1 \) this family of eigenfunctions also serves spectral problem in some indefinite inner product in the proper space of functions defined in the whole space \( \mathbb{R}^k \).