Solitons, Collapses and Turbulence

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Vortex structure at advanced stages of pumped 2d turbulence

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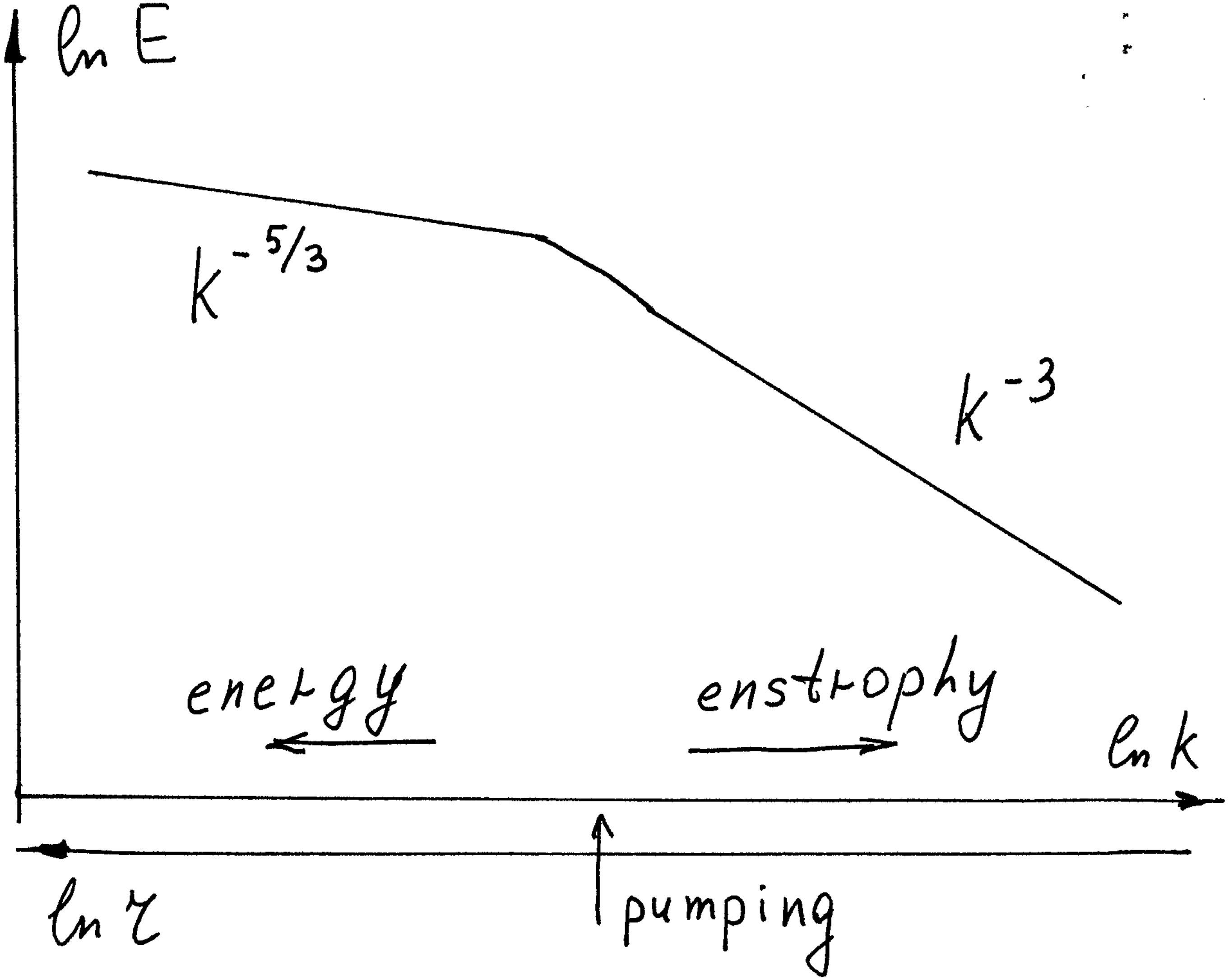
Two-dimensional turbulence – thin liquid films, soap films, atmosphere. Two quadratic integrals of motion – energy and enstrophy.

$$\int d^2r\ \boldsymbol{v}^2, \quad \int d^2r\ \omega^2, \quad \omega = \nabla \times \boldsymbol{v}.$$

Pumped turbulence – two cascades: enstrophy flows to small scales whereas energy flows to large scales (Kraichnan 1967, Leith 1968, Batchelor 1969).

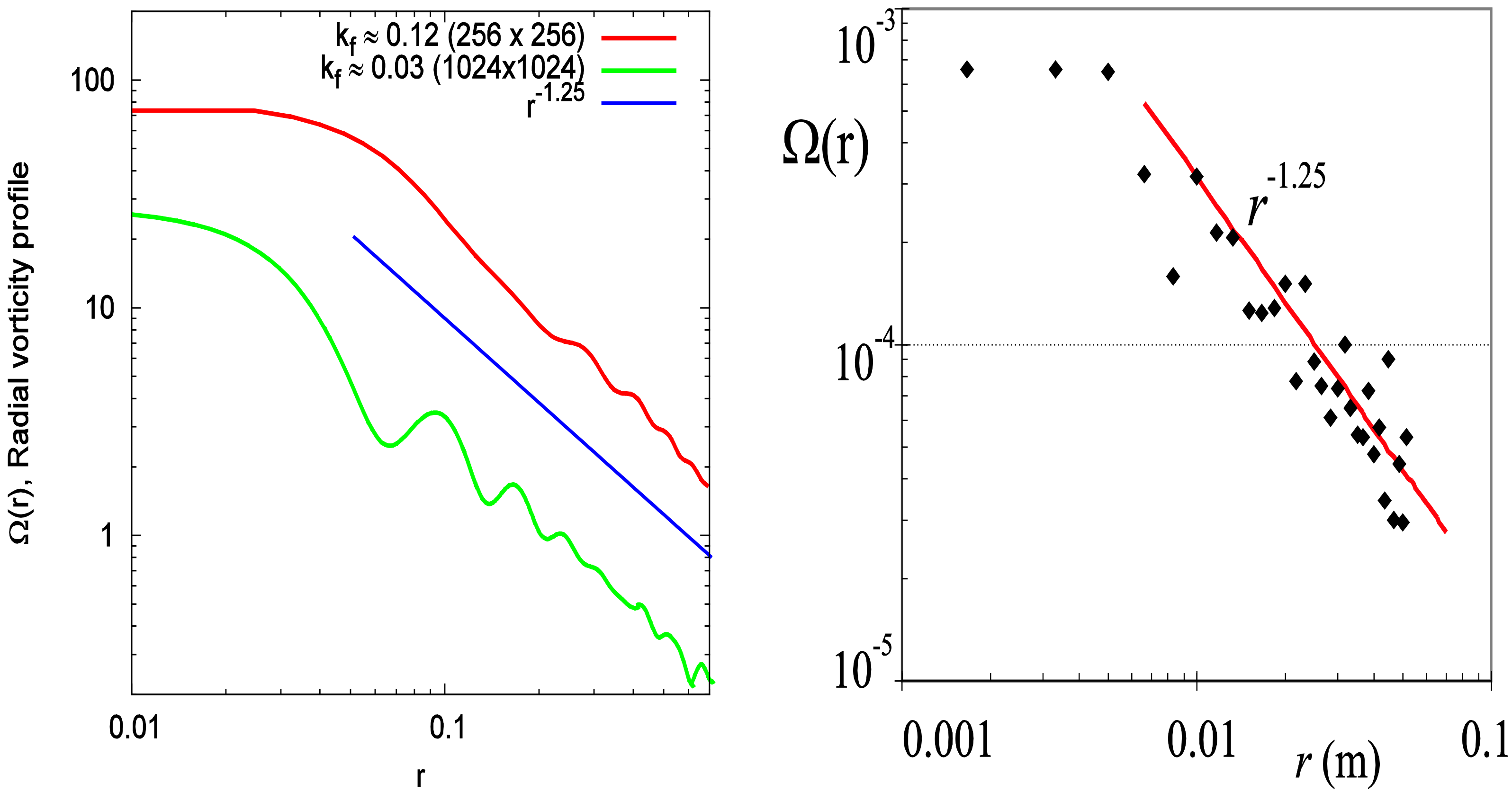

ln E
$k^{-5/3}$
k^{-3}
energy
enstrophy
ln k
ln r
pumping

Arguments a-la Kolmogorov lead to the spectrum $E(k) \propto k^{-3}$ for the direct (enstrophy) cascade and $E(k) \propto k^{-5/3}$ for the inverse (energy) cascade. Direct cascade – logarithmic correlation functions of vorticity (Falkovich, Lebedev 1994). Inverse cascade – an absence of anomalous scaling (?!) (Paret and Tabeling 1998, Boffetta, Celani and Vergassola 2000).

Weak dissipation at large scales leads the energy accumulation there thanks to the inverse cascade. Produces coherent structures. Experiment – single vortex (Shats, Xia, Punzmann and Falkovich 2007). Numerics (periodic conditions) – the vortex pair. (Chertkov, Connaughton, Kolokolov and Lebedev 2007). Application to atmosphere – Coriolis forces.

Coherent structures spontaneously appear if the scale where the inverse cascade should terminate (due to friction) is larger than the system size. Experiment – the vortex amplitude is determined by the wall friction. Numerics (frictionless) – the average velocity profile grows as $V \propto \sqrt{t}$, where t is time of the period when the pumping works.

Coherent structures – vortices with well-defined average velocity profile and relatively weak fluctuations on the background. Both, experiment and numerics, show that the vortices are isotropic and are characterized by power laws $V \propto r^{-1/4}$, $\Omega \propto r^{-5/4}$, where r is separation from the vortex center. Universality?



kf ≈ 0.12 (256 x 256)
kf ≈ 0.03 (1024x1024)
r-1.25
Ω(r), Radial vorticity profile
100
10
1
0.01
0.1
r
Ω(r)
r-1.25
10-3
10-4
10-5
0.001
0.01
0.1
r (m)

(A) (B)

The vorticity is a sum of the average component Ω and of the fluctuating components ω, $\langle\omega\rangle = 0$. Angular brackets mean averaging over times much larger than the characteristic turnover time and much smaller than the vortex evolution time t. The average and the fluctuating parts of the velocity are $\boldsymbol{V}$ and $\boldsymbol{v}$, respectively: $\Omega = \nabla \times \boldsymbol{V}$ and $\omega = \nabla \times \boldsymbol{v}$.

We assume that the pumping scale l is much larger than the viscous scale. Then the viscosity is irrelevant at scales larger than the pumping scale l. The viscousless equation for the vorticity is

$$\partial_t(\Omega + \omega) + (\boldsymbol{V} + \boldsymbol{v})\nabla(\Omega + \omega) = \phi.$$

Here ϕ is vorticity pumping, $\phi = \nabla \times \boldsymbol{f}$, where $\boldsymbol{f}$ is force per unit mass.

Separating the average and the fluctuating components, one finds

$$\partial_t \Omega + \boldsymbol{V} \nabla \Omega + \nabla \langle \boldsymbol{v} \omega \rangle = 0,$$
$$\partial_t \omega + \boldsymbol{V} \nabla \omega + \boldsymbol{v} \nabla \Omega + \boldsymbol{v} \nabla \omega - \nabla \langle \boldsymbol{v} \omega \rangle = \phi,$$

where we assumed $\langle \phi \rangle = 0$. The last term in the equation for Ω reflects the fluctuation contribution to the average equation, supporting the average profile.

Experiment and numerics show that the vorticity profile Ω inside the vortex is highly isotropic, and, moreover, is a scaling function of the separation from the vortex center: $\Omega \propto r^{-1-\eta}$ where η is an exponent to be determined. Then $\boldsymbol{V}$ has only polar component and $\boldsymbol{V}\nabla\Omega = 0$. Therefore

$$\partial_t \Omega + \nabla \langle \boldsymbol{v}\omega \rangle = 0.$$

The general relation $\omega = \nabla \times \boldsymbol{v}$ leads to the following relation between the vorticity and the radial velocity component v_r

$$\partial_\varphi \omega = -\frac{1}{r^2}\left(\partial_\varrho^2 + \partial_\varphi^2\right)(r v_r) \ ,$$

where $\varrho = \ln(r/L)$ and the incompressibility condition is

$$\partial_\varphi v_\varphi = -\frac{1}{r}\partial_\varrho(r v_r) = -(1 + \partial_\varrho) v_r \, .$$

The equation for Ω contains the object $\langle \omega v_r \rangle$ that can be expressed via the pair correlation function

$$\Phi(t, r_1, r_2, \varphi) = \langle v_r(t, r_1, \varphi_1) v_r(t, r_2, \varphi_2) \rangle \, ,$$

where $\varphi = \varphi_1 - \varphi_2$. The pair correlation function, as well as higher correlation functions, is a subject of investigation. Adiabaticity.

Because of the condition $\omega \ll \Omega$ the non-linear interaction of fluctuations is weak and can be neglected in the main approximation. Then we arrive at the following equation for the vorticity fluctuations

$$\partial_t \omega + \frac{V}{r} \partial_\varphi \omega + v_r \partial_r \Omega = \phi \,.$$

Then we can obtain equations for correlation functions.

For the pair correlation function

$$\left(\hat{\mathcal{N}}_1^{-1}\hat{\mathcal{K}}_1 - \hat{\mathcal{N}}_2^{-1}\hat{\mathcal{K}}_2\right)\Phi(r_1, r_2, \varphi) = 0\,,$$
$$\hat{\mathcal{N}} = (\partial_\varrho^2 + \partial_\varphi^2)r = r[(\partial_\varrho + 1)^2 + \partial_\varphi^2]\,,$$
$$\hat{\mathcal{K}} = V_0 \exp(-\eta\varrho)(\partial_\varrho^2 + 2\partial_\varrho + 2 + \partial_\varphi^2 - \eta^2)\,,$$

where we we omitted the pumping and the time derivative that are small inside the big vortex. Here V_0 is an average velocity at the vortex periphery $r \sim L$.

Note that zero modes Z_m of the operator $\widehat{\mathcal{K}}_m$ are power functions of r,

$$Z_m = \exp(im\varphi + \beta_m\varrho),$$

with exponents $\beta_m = -1 \pm \sqrt{m^2 + \eta^2 - 1}$. Further we take into account exponents

$$\beta_m = \sqrt{m^2 + \eta^2 - 1} - 1\,,$$

corresponding to suitable behavior of Z_m near origin.

We are looking for solutions that are analytic at close distances and angles. Note that the equation for Φ is homogeneous. We obtain an obvious solution

$$\Phi \propto Z_m(r_1, \varphi_1) Z_{-m}(r_2, \varphi_2)$$
$$+ Z_m(r_2, \varphi_2) Z_{-m}(r_1, \varphi_1) \propto r_1^{\beta_m} r_2^{\beta_m} \cos(m\varphi),$$

where Z_m is the zero mode of the operator $\hat{\mathcal{K}}_m$.

The next possible solution is

$$\Phi \propto X_m(\boldsymbol{r}_1)Z_{-m}(\boldsymbol{r}_2) + X_m(\boldsymbol{r}_2)Z_{-m}(\boldsymbol{r}_1) + Z_m(\boldsymbol{r}_1)X_{-m}(\boldsymbol{r}_2) + Z_m(\boldsymbol{r}_2)X_{-m}(\boldsymbol{r}_1).$$

Here the object X_m satisfies

$$X_m = \mathrm{exp}[im\varphi + (\beta_m + 1 + \eta)\varrho],$$

$$(\widehat{\mathcal{N}}_m)^{-1}\widehat{\mathcal{K}}_m X_m = A_m Z_m,$$

where A_m are real numbers.

The terms do not contribute to $\langle \omega v_r \rangle$ since they are symmetric in φ. All the admitted terms in the pair correlation function do not contribute to $\langle \omega v_r \rangle$. Thus we should find corrections to the pair correlation function related to the non-linear interaction of the fluctuations.

Then we arrive at the equation

$$\left(\widehat{\mathcal{N}}_1^{-1}\widehat{\mathcal{K}}_1 - \widehat{\mathcal{N}}_2^{-1}\widehat{\mathcal{K}}_2\right)\Phi(r_1, r_2, \varphi)$$
$$= r_1^2\widehat{\mathcal{N}}_1^{-1}\left\langle \boldsymbol{v}(r_1,\varphi_1)\nabla\omega(r_1,\varphi_1)v_r(r_2,\varphi_2)\right\rangle$$
$$-r_2^2\widehat{\mathcal{N}}_2^{-1}\left\langle v_r(r_1,\varphi_1)\boldsymbol{v}(r_2,\varphi_2)\nabla\omega(r_2,\varphi_2)\right\rangle \; .$$

We are interested in the correction $\delta\Phi$ that is a forced solution of the equation related to the third-order correlation function.

The third-order velocity correlation function is defined as

$$F = \langle v_r(t, r_1, \varphi_1) v_r(t, r_2, \varphi_2) v_r(t, r_3, \varphi_3)\rangle.$$

The correlation function satisfies

$$\frac{\partial}{\partial \varphi_1}(\widehat{\mathcal{N}}_1)^{-1}\widehat{\mathcal{K}}_1 F + \cdots + \frac{\partial}{\partial \varphi_n}(\widehat{\mathcal{N}}_3)^{-1}\widehat{\mathcal{K}}_3 F = 0,$$

subscripts mean variables r_1, r_2, r_3.

A simplest solution for the triple correlation function is

$$F \propto Z_m(\boldsymbol{r}_1) Z_k(\boldsymbol{r}_2) Z_{-m-k}(\boldsymbol{r}_3) + \text{permutations},$$

where permutations are produced over 1,2,3. However, it leads to a contribution to $\delta\Phi$ symmetric in φ that does not contribute to $\langle \omega v_r \rangle$.

The next solution can be constructed as follows

$$F = \alpha_m X_m(\boldsymbol{r}_1) Z_k(\boldsymbol{r}_2) Z_{-m-k}(\boldsymbol{r}_3) + \text{permutations}$$
$$+\alpha_k Z_m(\boldsymbol{r}_1) X_k(\boldsymbol{r}_2) Z_{-m-k}(\boldsymbol{r}_3) + \text{permutations}$$
$$+\alpha_{-k-m} Z_m(\boldsymbol{r}_1) Z_k(\boldsymbol{r}_2) X_{-m-k}(\boldsymbol{r}_3) + \text{permutations}.$$

The expression is a solution provided

$$\alpha_m m A_m + \alpha_k k A_k - \alpha_{-m-k}(m+k) A_{-m-k} = 0.$$

We should look for a solution with fastest grows to the center. The expression with the lowest power of r corresponds to $m = k = 1$, in the case

$$\delta\Phi \propto r^{4\eta+\sqrt{3+\eta^2}-2}.$$

Then we find

$$\langle v_r \omega \rangle \propto r^{-1} \delta\Phi \propto r^{4\eta+\sqrt{3+\eta^2}-3}.$$

Substituting the result into the equation for Ω and accounting for $\Omega \propto r^{-1-\eta}$ one obtains the equation

$$5\eta + \sqrt{3 + \eta^2} - 3 = 0.$$

A suitable solution of the equation is $\eta = 1/4$. It corresponds both to experiment and numerics.

Future developments. Anisotropy corrections. Coriolis forces. Passive scalar.