Building extended resolvent of heat operator via twisting transformations

M. Boiti, F. Pempinelli, A. Pogrebkov and B. Prinari

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KPII equation: $(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2}$

$u = u(x,t)$ is real, $x = (x_1, x_2)$

If $u_1(t, x_1)$ obeys KdV, then $u(t, x_1, x_2) = u_1(t, x_1 + \mu x_2 - 3\mu^2 t)$ solves KPII for an arbitrary constant $\mu \in \mathbb{R}$.

$$\mathcal{L}(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x) \quad \Rightarrow \quad \widetilde{\mathcal{L}}(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - \widetilde{u}(x)$$

$u(x)$: rapidly decaying $\Rightarrow$

$u_N(x)$: N-soliton potential, i.e., N solitons superimposed to the background potential $u(x)$ $\Rightarrow$

$\widetilde{u}(x) = u_N(x) + u'(x)$, $u'(x)$ is rapidly decaying.

$$\widetilde{\Phi} = \Phi_N + G_N u' \widetilde{\Phi}, \qquad e^{i\,\mathbf{k}(x_1' - x_1) + \mathbf{k}^2 (x_2' - x_2)} G_N(x, x', \mathbf{k}) \quad \text{must be bounded}$$

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$$\widetilde{\Phi} = \Phi_N + G_N u' \widetilde{\Phi}, \qquad e^{i\,\mathbf{k}(x_1' - x_1) + \mathbf{k}^2 (x_2' - x_2)} G_N(x, x', \mathbf{k}) \quad \text{must be bounded}$$

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$$\mathcal{L}(x,\partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x)$$

$$L(x,x';q) = \underbrace{\{-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2\}\delta(x-x')}_{L_0(x,x';q)} - u(x)\delta(x-x')$$

$$L(q)M(q) = M(q)L(q) = I$$

$$\left\{-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u(x)\right\}M(x,x';q) = \delta(x-x')$$

$$\left\{\partial_{x_2'} - q_2 + (\partial_{x_1'} - q_1)^2 - u(x')\right\}M(x,x';q) = \delta(x-x')$$

$$M(x,x';q) \in \mathcal{S}'(\mathbb{R}^6)$$

The Green's function:

$$G(x,x',\mathbf{k}) = e^{q(x-x')}M(x,x';q)\Big|_{q_1=\mathbf{k}_{\mathrm{Im}},\, q_2=\mathbf{k}_{\mathrm{Im}}^2 - \mathbf{k}_{\mathrm{Re}}^2}$$

Jost and dual Jost solutions of the original (decaying) potential:

$$(-\partial_{x_2} + \partial_{x_1}^2 - u(x))\Phi(x, \mathbf{k}) = 0, \qquad (\partial_{x_2} + \partial_{x_1}^2 - u(x))\Psi(x, \mathbf{k}) = 0$$

$$\mathbf{k} \in \mathbb{C}$$

$$\overline{\Phi(x, \mathbf{k})} = \Phi(x, -\overline{\mathbf{k}}), \qquad \overline{\Psi(x, \mathbf{k})} = \Psi(x, -\overline{\mathbf{k}})$$

Normalization condition at $\mathbf{k}$-infinity

$$\lim_{\mathbf{k}\to\infty} e^{i\,\mathbf{k}\,x_1 + \mathbf{k}^2\,x_2}\Phi(x, \mathbf{k}) = 1, \qquad \lim_{\mathbf{k}\to\infty} e^{-i\,\mathbf{k}\,x_1 - \mathbf{k}^2\,x_2}\Psi(x, \mathbf{k}) = 1$$

$$u(x) = -2i \lim_{\mathbf{k}\to\infty} \mathbf{k}\,\partial_{x_1}\left(e^{i\,\mathbf{k}\,x_1 + \mathbf{k}^2\,x_2}\Phi(x, \mathbf{k})\right) = 2i \lim_{\mathbf{k}\to\infty} \mathbf{k}\,\partial_{x_1}\left(e^{-i\,\mathbf{k}\,x_1 - \mathbf{k}^2\,x_2}\Psi(x, \mathbf{k})\right)$$

Resolvent:

$$M(x, x'; q) = -\operatorname{sgn}(x_2 - x_2')\frac{e^{q(x'-x)}}{2\pi}\times$$
$$\times \int\limits_{\mathbf{k}_{\mathrm{Im}}=q_1} d\,\mathbf{k}_{\mathrm{Re}}\,\theta\left((q_2 + \operatorname{Re}(\mathbf{k}^2))(x_2 - x_2')\right)\Phi(x, \mathbf{k})\Psi(x', \mathbf{k})$$

Darboux transformation via "twisting" L to L'

$$L = L_0 - u \quad \Rightarrow \quad L' = L_0 - u' : \qquad L'\zeta = \zeta L, \qquad \eta L' = L\eta$$

$$\overline{\zeta(x, x'; q_1)} = \zeta(x, x'; q_1), \qquad \overline{\eta} = \eta, \qquad \eta\zeta = I, \quad \text{then} \quad L = \eta L'\zeta$$

$$P \stackrel{\text{def}}{=} I - \zeta\eta, \quad \text{then} \quad P^2 = P, \qquad \overline{P} = P, \qquad P\zeta = \eta P = 0$$

New Jost solutions are given as

$$\Phi'(x, \mathbf{k}) = \int dy\, e^{\mathbf{k}_{\mathrm{Im}}(x_1 - y_1)} \zeta(x, y; \mathbf{k}_{\mathrm{Im}})\, \Phi(y, \mathbf{k})$$

$$\Psi'(x, \mathbf{k}) = \int dy\, \Psi(y, \mathbf{k})\, e^{\mathbf{k}_{\mathrm{Im}}(y_1 - x_1)} \eta(y, x; \mathbf{k}_{\mathrm{Im}})$$

Conditions: self-conjugation, asymptotic behavior and existence of poles of $\Phi'(x, \mathbf{k})$ and $\Psi'(x, \mathbf{k})$:

$$\Phi'_{b_l}(x) = \operatorname*{res}_{\mathbf{k}=ib_l} \Phi'(x, \mathbf{k})$$

$$\Psi'_{a_j}(x) = \operatorname*{res}_{\mathbf{k}=ia_j} \Psi'(x, \mathbf{k})$$

$a_1, \ldots, a_{N_a}$, $b_1, \ldots, b_{N_b}$ are $N_a + N_b$ parameters, all different; $a_j, b_l \in \mathbb{R}$

Solution is possible only if $N_a, N_b \geq 1$.

$$N_b \times N_a\text{-matrix: } \mathcal{F}(x) = \|\, \mathcal{F}_{lj}(x)\|_{l=1,\dots,N_b}^{j=1,\dots,N_a}$$

$$\mathcal{F}_{lj}(x) \stackrel{\text{def}}{=} \mathcal{F}(x, ib_l, ia_j) \qquad \mathcal{F}(x, \mathbf{k}, \mathbf{k}') \stackrel{\text{def}}{=} \int\limits_{(\mathbf{k}_{\text{Im}} - \mathbf{k}'_{\text{Im}})\infty}^{x_1} dx'_1 \Psi(x', \mathbf{k})\Phi(x', \mathbf{k}')\Big|_{x'_2 = x_2}$$

The constant $N_a \times N_b$ matrix c, the unity $N_a \times N_a$ matrix E_{N_a} and the unity $N_b \times N_b$ matrix E_{N_b}

$$u'(x) = u(x) - 2\partial_{x_1}^2 \ln\det(E_{N_b} + \mathcal{F}\, c) \equiv u(x) - 2\partial_{x_1}^2 \ln\det(E_{N_a} + c\,\mathcal{F})$$

$$\Phi'(x, \mathbf{k}) = \Phi(x, \mathbf{k}) - \sum_{j=1}^{N_a} \sum_{l=1}^{N_b} \Phi(x, ia_j) \left(\left(E_{N_a} + c\,\mathcal{F}(x) \right)^{-1} c \right)_{jl} \mathcal{F}(x, ib_l, \mathbf{k})$$

$$\Psi'(x, \mathbf{k}) = \Psi(x, \mathbf{k}) - \sum_{j=1}^{N_a} \sum_{l'=1}^{N_b} \mathcal{F}(x, \mathbf{k}, ia_j) \left(c \left(E_{N_b} + \mathcal{F}(x) c \right)^{-1} \right)_{jl} \Psi(x, ib_l).$$

so that for residuals we have:

$$\Phi'_{b_l}(x) = -i \sum_{j=1}^{N_a} \Phi'(x, ia_j) c_{jl}, \qquad \Psi'_{a_j}(x) = i \sum_{l=1}^{N_b} c_{jl} \Psi'(x, ib_l)$$

Let us, now, use the parameters

$$\{\alpha_1, \ldots, \alpha_{\mathcal{N}}\} = \{a_1, \ldots, a_{N_a}, b_1 \ldots, b_{N_b}\}, \quad \text{where } \mathcal{N} = N_a + N_b$$

$$P(x, x'; q) = i e^{q_1(x_1' - x_1)} \delta(x_2 - x_2') \sum_{n=1}^{\mathcal{N}} \theta(q_1 - \alpha_n) \operatorname*{res}_{\mathbf{k}=i\alpha_n} \Phi'(x, \mathbf{k}) \Psi'(x', \mathbf{k})$$

Resolvent. We have seen that $L = \eta L' \zeta$. But

$$L' = \zeta L \eta + L_\Delta, \quad \text{where} \quad L_\Delta = L'P = PL'$$

Correspondingly, we write for the resolvent $M' = \zeta M \eta + M_\Delta$

$$(\zeta M \eta)(x, x'; q) = -\operatorname{sgn}(x_2 - x_2') \frac{e^{q(x'-x)}}{2\pi} \times$$
$$\times \int\limits_{\mathbf{k}_{\mathrm{Im}} = q_1} d\,\mathbf{k}_{\mathrm{Re}}\, \theta\left((q_2 + \mathrm{Re}(\mathbf{k}^2))(x_2 - x_2')\right) \Phi'(x, \mathbf{k}) \Psi'(x', \mathbf{k})$$

$$L' \zeta M \eta = I - P$$

M_Δ has to obey $L' M_\Delta = M_\Delta L' = P$

Formally

$$M_\Delta(x,x';q) = \mp i e^{q(x'-x)}\theta\big(\pm(x_2-x_2')\big)\sum_{n=1}^{\mathcal{N}}\theta(q_1-\alpha_n)\operatorname*{res}_{\mathbf{k}=i\alpha_n}\Phi'(x,\mathbf{k})\Psi'(x',\mathbf{k})$$

$$M_\Delta(x,x';q)\in\mathcal{S}(\mathbb{R}^6) \quad ?$$

Pure N-soliton potential

In the case $u(x)\equiv 0$ one gets the general N-soliton potential, where $N=\max\{N_a,N_b\}$.

$$\mathcal{F}(x,\mathbf{k},\mathbf{k}') = i\frac{e^{i(\mathbf{k}-\mathbf{k}')x_1+(\mathbf{k}^2-\mathbf{k}'^2)x_2}}{\mathbf{k}'-\mathbf{k}}$$

$$\Phi'(x,\mathbf{k})\to\left(\prod_{j=1}^{N_a}(a_j+i\,\mathbf{k})\right)\Phi'(x,\mathbf{k}),\qquad \Psi'(x,\mathbf{k})\to\left(\prod_{j=1}^{N_a}(a_j+i\,\mathbf{k})\right)^{-1}\Psi'(x,\mathbf{k})$$

so that $\Phi'(x,\mathbf{k})$ has poles in all points α_n, $n=1,\ldots,\mathcal{N}$ and $\Psi'(x,\mathbf{k})$ is analytic.

$$\Phi'(x,\mathbf{k}) = \frac{\tau_\chi(x,\mathbf{k})}{\tau(x)}e^{-i\,\mathbf{k}\,x_1-\mathbf{k}^2\,x_2},\qquad \Psi'(x,\mathbf{k}) = \frac{\tau_\xi(x,\mathbf{k})}{\tau(x)}e^{i\,\mathbf{k}\,x_1+\mathbf{k}^2\,x_2}$$

$$u'(x) = -2\partial_{x_1}^2\log\tau(x)$$

$$\tau_\chi(x,\mathbf{k}) = \det\mathcal{D}\; e^{-A(x)}(\alpha+i\,\mathbf{k})^{-1}\widetilde{\mathfrak{A}},\quad \tau'_\xi(x,\mathbf{k}) = \det\mathcal{D}\; e^{-A(x)}(\alpha+i\,\mathbf{k})\widetilde{\mathfrak{A}}$$

and

$$\tau(x) = \det\mathcal{D}\; e^{-A(x)}\widetilde{\mathfrak{A}}$$

where $\mathcal{D}$ is $N_a\times\mathcal{N}$-matrix, $\mathcal{D} = \big(E_{N_a}, c'\big)$, $\mathcal{N} = N_a + N_b$,

$$\widetilde{\mathfrak{A}} = \|(-\alpha_n)^{N_a-j}\|_{nj},\quad n = 1,\ldots,\mathcal{N},\quad j = 1,\ldots,N_a$$

$$e^{-A(x)} = \operatorname{diag}\{e^{-A_1(x)},\ldots,e^{-A_{\mathcal{N}}(x)}\},\quad A_n(x) = \alpha_n x_1 + \alpha_n^2 x_2$$

$$\tau_\chi(x,\mathbf{k}) = \sum_{1\le n_1<n_2<\cdots<n_{N_a}\le\mathcal{N}} f_{n_1,\ldots,n_{N_a}} \prod_{j=1}^{N_a} \frac{e^{-A_{n_j}(x)}}{\alpha_{n_j}+i\,\mathbf{k}}$$

$$\tau_\xi(x,\mathbf{k}) = \sum_{1\le n_1<n_2<\cdots<n_{N_a}\le\mathcal{N}} f_{n_1,\ldots,n_{N_a}} \prod_{j=1}^{N_a} (\alpha_{n_j}+i\,\mathbf{k})e^{-A_{n_j}(x)}$$

$$\tau(x) = \sum_{1\le n_1<n_2<\cdots<n_{N_a}\le\mathcal{N}} f_{n_1,\ldots,n_{N_a}} \prod_{j=1}^{N_a} e^{-A_{n_j}(x)}$$

$$f_{n_1,n_2,\ldots,n_{N_a}} = W(-\alpha_{n_1},\ldots,-\alpha_{n_{N_a}})\,\mathcal{D}'(n_1,\ldots,n_{N_a}) \ge 0$$

$$\tau_\chi(x,\mathbf{k}) = \det \mathcal{D}\ e^{-A(x)}(\alpha + i\,\mathbf{k})^{-1}\widetilde{\mathfrak{A}}, \quad \tau'_\xi(x,\mathbf{k}) = \det \mathcal{D}\ e^{-A(x)}(\alpha + i\,\mathbf{k})\widetilde{\mathfrak{A}}$$

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$$e^{-A(x)} = \mathrm{diag}\{e^{-A_1(x)}, \ldots, e^{-A_{\mathcal{N}}(x)}\}, \quad A_n(x) = \alpha_n x_1 + \alpha_n^2 x_2$$

$$\tau_\chi(x,\mathbf{k}) = \sum_{1\le n_1 < n_2 < \cdots < n_{N_a} \le \mathcal{N}} f_{n_1,\ldots,n_{N_a}} \prod_{j=1}^{N_a} \frac{e^{-A_{n_j}(x)}}{\alpha_{n_j} + i\,\mathbf{k}}$$

$$\tau_\xi(x,\mathbf{k}) = \sum_{1\le n_1 < n_2 < \cdots < n_{N_a} \le \mathcal{N}} f_{n_1,\ldots,n_{N_a}} \prod_{j=1}^{N_a} (\alpha_{n_j} + i\,\mathbf{k}) e^{-A_{n_j}(x)}$$

$$\tau(x) = \sum_{1\le n_1 < n_2 < \cdots < n_{N_a} \le \mathcal{N}} f_{n_1,\ldots,n_{N_a}} \prod_{j=1}^{N_a} e^{-A_{n_j}(x)}$$

$$f_{n_1,n_2,\ldots,n_{N_a}} = W(-\alpha_{n_1}, \ldots, -\alpha_{n_{N_a}})\,\mathcal{D}'(n_1, \ldots, n_{N_a}) > 0$$

$$\alpha_1 < \cdots < \alpha_{\mathcal{N}}$$

Let us start with the one-dimensional situation:

$$(-\partial_{x_1}^2 + u_1(x_1))\varphi(x_1, \mathbf{k}) = \mathbf{k}^2\,\varphi(x_1, \mathbf{k})$$

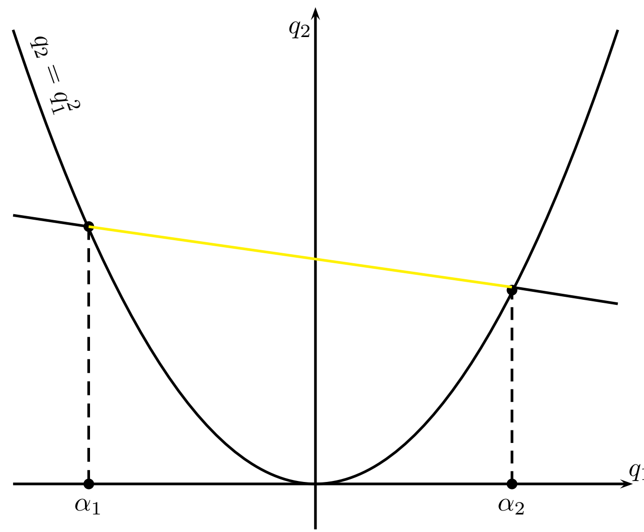
Let $i\alpha$ $(\alpha > 0)$ be an eigenvalue and $\varphi_\alpha(x_1)$ be residual of $\varphi(x_1, \mathbf{k})$. Asymptotically $\varphi(x_1, \mathbf{k}) \sim e^{-\alpha|x_1|}$. In other words, $e^{-q_1 x_1}\varphi(x_1, \mathbf{k})$ is bounded for any x_1 if $|q_1| < \alpha$. Then equation $(-\partial_{x_2} + \partial_{x_1}^2 - u_1(x_1))\Phi_\alpha(x) = 0$ has solution $\Phi_\alpha(x) = e^{\alpha^2 x_2}\varphi_\alpha(x_1)$. Thus $e^{-q_1 x_1 - q_2 x_2}\Phi_\alpha(x)$ is bounded for any $x = (x_1, x_2)$ only if $|q_1| < \alpha$ and $q_2 = \alpha^2$.

Performing transformation $u_1(x_1) \to u_1(x_1 + (\alpha_1 + \alpha_2)x_2)$ $(\alpha_1 < \alpha_2$ are some real parameters) and denoting $\alpha = \dfrac{\alpha_2 - \alpha_1}{2}$, we get that equation $(-\partial_{x_2} + \partial_{x_1}^2 - u_1(x_1 + (\alpha_1 + \alpha_2)x_2))\Phi(x) = 0$ has solution such that $e^{-q_1 x_1 - q_2 x_2}\Phi(x)$ is bounded for all x iff $\alpha_1 < q_1 < \alpha_2$, $q_{12} = 0$, where we introduce

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n, \quad m, n = 1, \ldots, \mathcal{N}$$

$$\left[-\partial_{x_2}+\partial_{x_1}^2-u_1(x_1+(\alpha_1+\alpha_2)x_2)\right]\Phi(x)=0$$
$$\left[-\partial_{x_2}-q_2+(\partial_{x_1}+q_1)^2-u_1(x_1+(\alpha_1+\alpha_2)x_2)\right]e^{-qx}\Phi(x)=0$$

$e^{-qx}\Phi(x)$ is bounded for all x
iff $\alpha_1<q_1<\alpha_2$, $q_{12}=0$



$$q_{mn}=q_2-(\alpha_m+\alpha_n)q_1+\alpha_m\alpha_n\equiv$$
$$\equiv q_2-q_1^2+(q_1-\alpha_m)(q_1-\alpha_n)$$

$$G(x,x',\mathbf{k})=e^{q(x-x')}M(x,x';q)\Big|_{q_1=\mathbf{k}_{\mathrm{Im}},\,q_2=\mathbf{k}_{\mathrm{Im}}^2-\mathbf{k}_{\mathrm{Re}}^2},\quad q_2-q_1^2=-\mathbf{k}_{\mathrm{Re}}^2$$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$
$$\left[-\partial_{x_2} + \partial_{x_1}^2 - u'(x)\right]\Phi'_m(x) = 0$$
$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right] e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

$$\kappa_m(q) = \left(\prod_{k=m}^{N_b} \theta(q_{k,k+N_a})\right)\left(\prod_{k=1}^{m} \theta(-q_{k,k+N_b})\right)$$

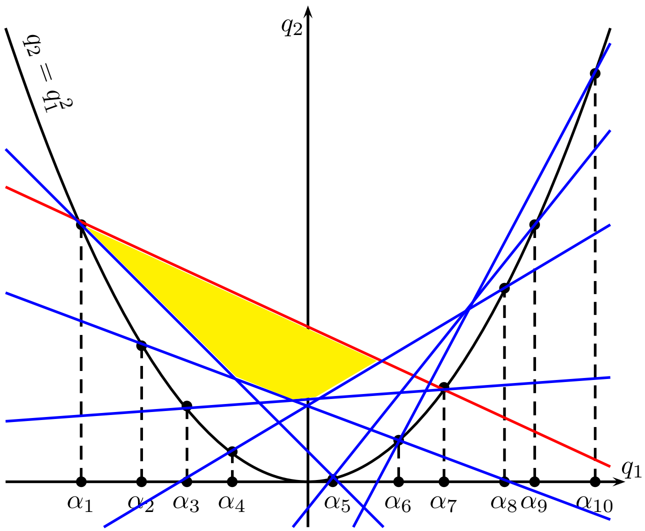
in the case $1 \le m \le N_a < N_b$

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$
$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$

$u'(x)$ has rays:

$x_2 \to +\infty$: $\quad x_1 + (\alpha_k + \alpha_{k+N_a})x_2 = \text{const}$, $k = 1, \dots, N_b$,

$x_2 \to -\infty$: $\quad x_1 + (\alpha_k + \alpha_{k+N_b})x_2 = \text{const}$, $k = 1, \dots, N_a$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 1$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$

$$\left[-\partial_{x_2} + \partial_{x_1}^2 - u'(x)\right]\Phi'_m(x) = 0$$

$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right] e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

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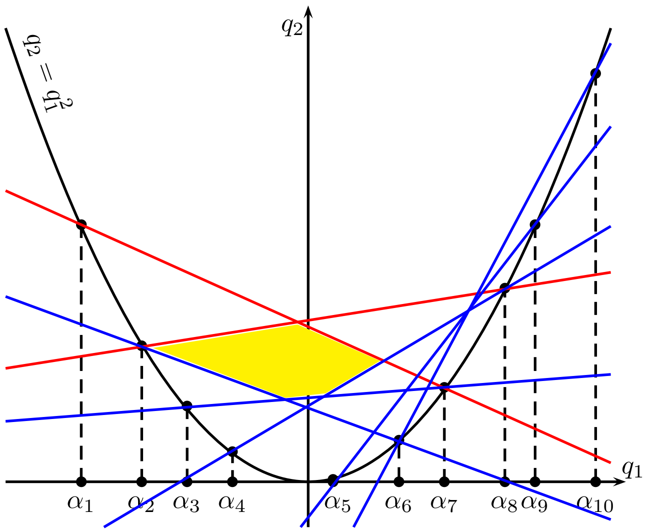
$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$
$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 2$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$

$$\left[-\partial_{x_2} + \partial_{x_1}^2 - u'(x)\right]\Phi'_m(x) = 0$$

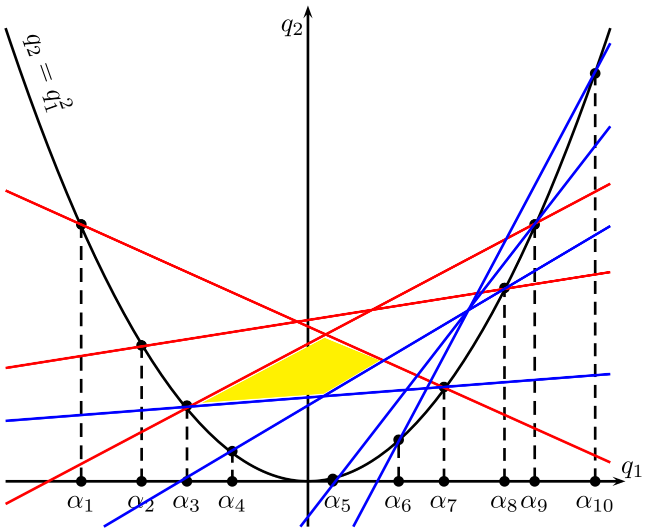
$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right]e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

$$\kappa_m(q) = \left(\prod_{k=m}^{N_b} \theta(q_{k,k+N_a})\right)\left(\prod_{k=1}^{m} \theta(-q_{k,k+N_b})\right)$$

in the case $1 \le m \le N_a < N_b$

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$
$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 3$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$

$$\left[-\partial_{x_2} + \partial_{x_1}^2 - u'(x)\right]\Phi'_m(x) = 0$$

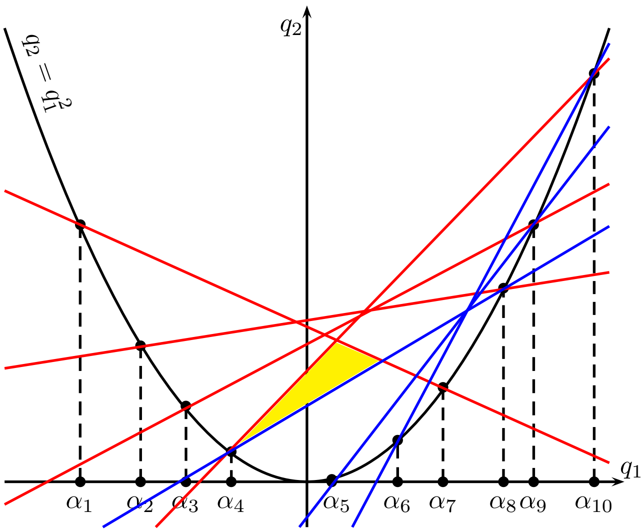
$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right]e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

$$\kappa_m(q) = \left(\prod_{k=m}^{N_b} \theta(q_{k,k+N_a})\right)\left(\prod_{k=1}^{m} \theta(-q_{k,k+N_b})\right)$$

in the case $1 \le m \le N_a < N_b$

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$
$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 4$

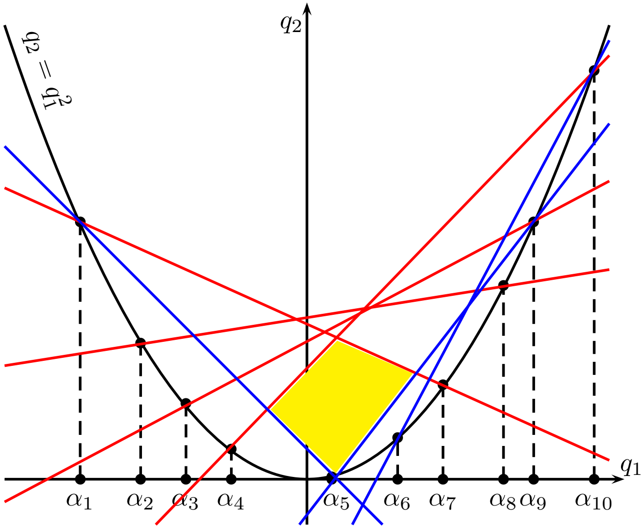
$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$
$$\left[-\partial_{x_2} + \partial_{x_1}^2 - u'(x)\right]\Phi'_m(x) = 0$$
$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right]e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

$$\kappa_m(q) = \left(\prod_{k=m}^{N_b} \theta(q_{k,k+N_a})\right)\left(\prod_{k=1}^{m-N_a} \theta(q_{k,k+N_a})\right) \times$$
$$\times \left(\prod_{k=1}^{N_a} \theta(-q_{k,k+N_b})\right)$$

in the case $N_a + 1 \leq m \leq N_b$

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$
$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 5$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$

$$\left[-\partial_{x_2} + \partial^2_{x_1} - u'(x)\right]\Phi'_m(x) = 0$$

$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right]e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

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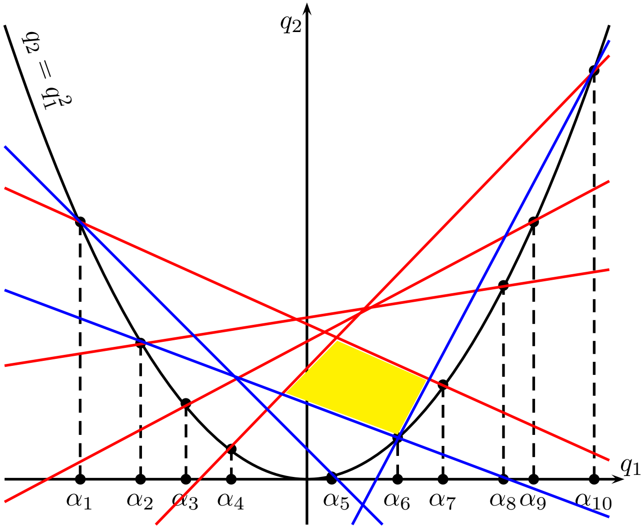
$$\times\left(\prod_{k=1}^{N_a}\theta(-q_{k,k+N_b})\right)$$

in the case $N_a + 1 \le m \le N_b$

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$
$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 6$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$

$$\left[-\partial_{x_2} + \partial^2_{x_1} - u'(x)\right]\Phi'_m(x) = 0$$

$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right] e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

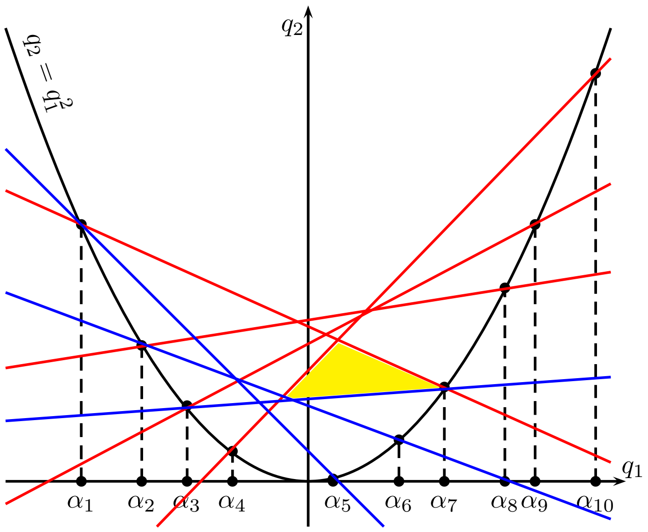
$$\kappa_m(q) = \left(\prod_{k=1}^{m-N_a} \theta(q_{k,k+N_a})\right) \times$$

$$\times \left(\prod_{k=m-N_b}^{N_a} \theta(-q_{k,k+N_b})\right)$$

in the case $N_b + 1 \le m \le \mathcal{N} = N_a + N_b$

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$

$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 7$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$

$$\left[-\partial_{x_2} + \partial_{x_1}^2 - u'(x)\right]\Phi'_m(x) = 0$$

$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right]e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

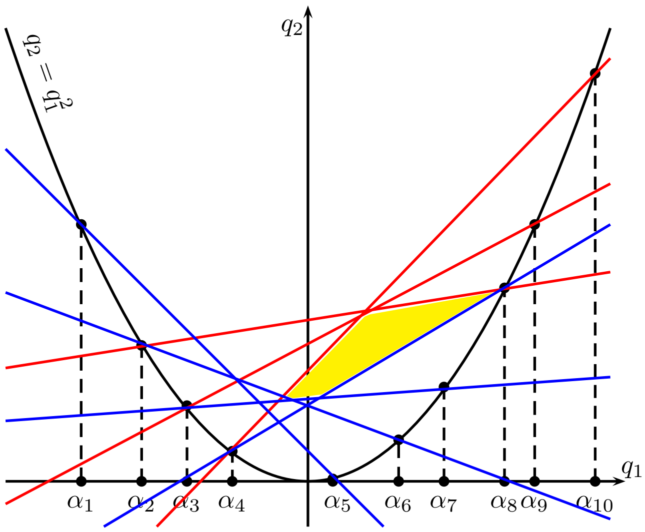
$$\kappa_m(q) = \left(\prod_{k=1}^{m-N_a}\theta(q_{k,k+N_a})\right)\times$$

$$\times\left(\prod_{k=m-N_b}^{N_a}\theta(-q_{k,k+N_b})\right)$$

in the case $N_b + 1 \leq m \leq \mathcal{N} = N_a + N_b$

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$

$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4$, $N_b = 6$, $\mathcal{N} = 10$, $m = 8$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$

$$\left[-\partial_{x_2} + \partial^2_{x_1} - u'(x)\right]\Phi'_m(x) = 0$$

$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right]e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

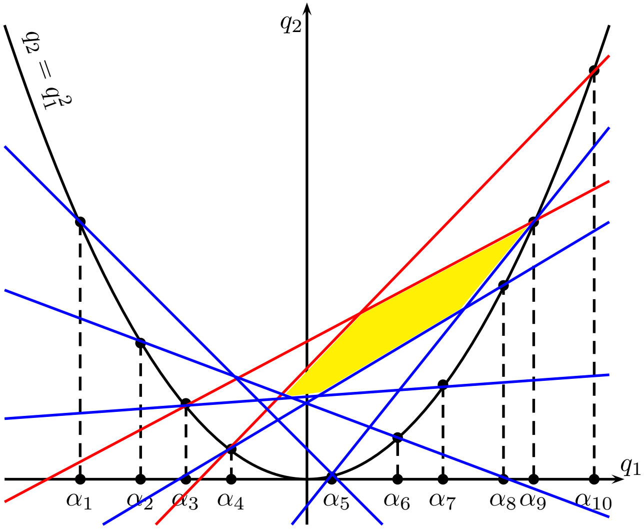
$$\kappa_m(q) = \left(\prod_{k=1}^{m-N_a} \theta(q_{k,k+N_a})\right) \times$$

$$\times \left(\prod_{k=m-N_b}^{N_a} \theta(-q_{k,k+N_b})\right)$$

in the case $N_b + 1 \le m \le \mathcal{N} = N_a + N_b$

$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$

$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 9$

$$\Phi'_m(x) = \operatorname*{res}_{\mathbf{k}=i\alpha_m} \Phi'(x,\mathbf{k})$$

$$\left[-\partial_{x_2} + \partial^2_{x_1} - u'(x)\right]\Phi'_m(x) = 0$$

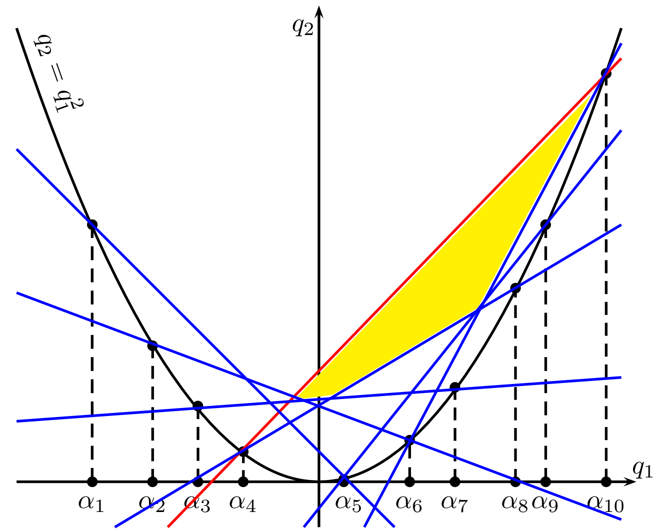
$$\left[-\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u'(x)\right]e^{-qx}\Phi'_m(x) = 0$$

$\kappa_m(q)e^{-qx}\Phi'_m(x)$ is exponentially decaying for all x iff q belongs to the polygon:

$$\kappa_m(q) = \left(\prod_{k=1}^{m-N_a} \theta(q_{k,k+N_a})\right) \times \left(\prod_{k=m-N_b}^{N_a} \theta(-q_{k,k+N_b})\right)$$

in the case $N_b + 1 \le m \le \mathcal{N} = N_a + N_b$

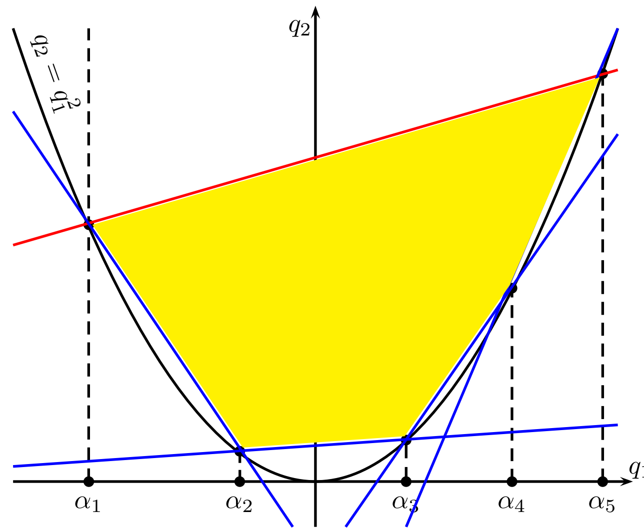
$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv \equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$



$N_a = 4,\ N_b = 6,\ \mathcal{N} = 10,\ m = 10$

$$N_a = 1,\ N_b = 4,\ \mathcal{N} = 5,\ m = 1.$$

$$\kappa_m(q) = \left(\prod_{k=m}^{N_b} \theta(q_{k,k+N_a})\right)\left(\prod_{k=1}^{m} \theta(-q_{k,k+N_b})\right)$$



$$q_{mn} = q_2 - (\alpha_m + \alpha_n)q_1 + \alpha_m\alpha_n \equiv$$
$$\equiv q_2 - q_1^2 + (q_1 - \alpha_m)(q_1 - \alpha_n)$$

$$M_\Delta(x, x'; q) = \begin{cases} -ie^{q(x'-x)}\theta(x_2 - x_2') \sum_{n=1}^{\mathcal{N}} \theta(q_1 - \alpha_n)\Phi'_{\alpha_n}(x)\Psi'(x', i\alpha_n), & \text{above} \\ +ie^{q(x'-x)}\theta(x_2' - x_2) \sum_{n=1}^{\mathcal{N}} \theta(q_1 - \alpha_n)\Phi'_{\alpha_n}(x)\Psi'(x', i\alpha_n), & \text{below} \end{cases}$$