1. Whether the Camassa-Holm equation describes shallow water waves?
2. Reductions in continuous and discrete integrable systems

A.Mikhailov
Based on results with (1) R.Bhatt,
(2) R.Bury and P.Xenitidis

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Chernogolovka

R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 16611664.

A. Constantin, D. Lannes, The Hydrodynamical Relevance of the Camassa Holm and Degasperis Procesi Arch. Rational Mech. Anal. 192 (2009) 165186

H.R. Dullin, G. Gottwald, D.D. Holm, An integrable shallow water equation with linear and nonlinear dispersion, Phys. Rev. Lett. 87 (2001) 45014504.

H.R. Dullin, G.A. Gottwald, D.D. Holm, On asymptotically equivalent shallow water wave equations Physica D 190 (2004) 1-14.

A.S. Fokas, Q.M. Liu, Asymptotic integrability of water waves, Phys. Rev. Lett. 77 (1996) 23472351.

A.S. Fokas, On a class of physically important integrable equations, Physica D 87 (1995) 145150.

A.S. Fokas, B. Fuchssteiner, Bcklund transformations for hereditary symmetries, Nonlinear Anal. TMA 5 (1981) 423432.

R.S. Johnson, Camassa–Holm, Korteweg–de Vries and related models for water waves, J. Fluid Mech. 455 (2002) 6382.

Shallow water waves asymptotic $\epsilon = \frac{h_0^2}{l^2}$ and $\mu \geq \epsilon > \mu^2$

$$\begin{aligned} &x \mapsto lx, \qquad z \mapsto h_0 z, \qquad t \mapsto \frac{l}{c_0} t, \\ &\eta \mapsto \mu h_0 \eta, \quad \phi \mapsto \mu l c_0 \phi, \quad c_0 = \sqrt{g h_0}. \end{aligned}$$

$$\begin{aligned} &\epsilon \phi_{xx} + \phi_{zz} = 0 \quad \text{and} \quad \phi_z = 0 \text{ at } z = 0, \\ &\frac{1}{\epsilon} \phi_z = \mu \eta_x \phi_x + \eta_t \qquad \text{at } z = 1 + \mu \eta(x,t), \\ &\phi_t + \frac{1}{2} \left(\mu \phi_x^2 + \frac{\mu}{\epsilon} \phi_z^2 \right) + \eta = 0 \qquad \text{at } z = 1 + \mu \eta(x,t). \end{aligned}$$

$$\phi = F - \frac{\epsilon}{2} z^2 F_{xx} + \frac{\epsilon^2}{24} z^4 F_{xxxx} - \frac{\epsilon^3}{720} z^6 F_{xxxxxx} + O(\epsilon^4), \qquad w(x,t) = F_x$$

$$\begin{aligned}
0 &= \eta_t + w_x + (\eta_x w + \eta\, w_x)\,\mu - \frac{\epsilon}{6} w_{x,x,x} \\
&- \frac{\mu\epsilon}{2}(\eta_x w_{x,x} + w_{x,x,x}\eta) + \frac{\epsilon^2}{120} w_{x,x,x,x,x} \\
&- \frac{\mu^2\epsilon}{2}\left(2\eta_x w_{x,x}\eta + w_{x,x,x}\eta^2\right) + \frac{\mu\epsilon^2}{24}(w_{x,x,x,x,x}\eta + \eta_x w_{x,x,x,x}) - \frac{\epsilon^3}{5040} w_{x,x,x,x,x,x,x} + \cdot
\end{aligned}$$

$$\begin{aligned}
0 &= w_t + \eta_x + \mu\, w w_x - \frac{\epsilon}{2}(w_{t,x,x} + 2\sigma\,\eta_{x,x,x}) \\
&- \frac{\mu\epsilon}{2}(2 w_{t,x}\eta_x + 2 w_{t,x,x}\eta + w w_{x,x,x} - w_x w_{x,x}) + \frac{\epsilon^2}{24} w_{t,x,x,x,x} \\
&+ \mu^2\epsilon\left({w_x}^2\eta_x - 1/2\, w_{t,x,x}\eta^2 + w_x w_{x,x}\eta - w w_{x,x,x}\eta - w w_{x,x}\eta_x - w_{t,x}\eta\,\eta_x\right) \\
&+ \frac{\mu\epsilon^2}{24}(4 w_{t,x,x,x,x}\eta + 2 w_{x,x} w_{x,x,x} - 3 w_x w_{x,x,x,x} + w_{x,x,x,x,x} w + 4 w_{t,x,x,x}\eta_x) \\
&- \frac{\epsilon^3}{720} w_{t,x,x,x,x,x,x} + \cdots
\end{aligned}$$

Reduction to unidirectional waves

$$
\begin{aligned}
w = \quad & \eta + \mu\, a\, \eta^2 + \epsilon\, b\, \eta_{x,x} + \mu^2\, c\, \eta^3 + \mu\, \epsilon \left(d_1\, \eta_x^2 + d_2\, \eta\eta_{x,x}\right) + \epsilon^2\, e\, \eta_{x,x,x,x} \\
+ \quad & \mu^3\, f\, \eta^4 + \mu^2 \epsilon \left(g_1\, \eta^2 \eta_{x,x} + g_2\, \eta\eta_x^2 + g_3 D_x^{-1}(\eta_x^3)\right) \\
+ \quad & \mu\epsilon^2 \left(h_1\, \eta\eta_{x,x,x,x} + h_2 \eta_x \eta_{x,x,x} + h_3\, \eta_{x,x}^2\right) + \epsilon^3\, i\, \eta_{x,x,x,x,x,x}
\end{aligned}
$$

$$
\begin{aligned}
0 = \quad & \eta_x + \eta_t + \frac{3}{2}\, \mu\, \eta_x \eta + \frac{\epsilon}{6}\, \eta_{x,x,x} \\
& -\frac{3}{8}\, \eta^2 \eta_x \mu^2 + \left(\frac{5}{12}\, \eta\, \eta_{x,x,x} + \frac{23}{24}\, \eta_x \eta_{x,x}\right) \epsilon\, \mu + \epsilon^2 \frac{19}{360}\, \eta_{x,x,x,x,x} \\
& + \left(\frac{23}{16} \eta\, \eta_x \eta_{x,x} + \frac{5}{16} \eta^2 \eta_{x,x,x} + \frac{19}{32} {\eta_x}^3\right) \epsilon\, \mu^2 \\
& + \left(\frac{1079}{1440}\, \eta_{x,x,x,x} \eta_x + \frac{317}{288}\, \eta_{x,x} \eta_{x,x,x} + \frac{19}{80}\, \eta_{x,x,x,x,x} \eta\right) \epsilon^2 \mu \\
& + \left(\frac{55}{3024}\, \eta_{x,x,x,x,x,x,x}\right) \epsilon^3 + \frac{3}{16}\, \eta^3 \eta_x \mu^3 + \mathcal{O}(\mu^4, \mu^3\epsilon, \mu^2\epsilon^2, \mu\epsilon^3, \epsilon^4)
\end{aligned}
$$

Making a Kodama Transformation

$$\eta\,(x,t) = u + \mu\,(\alpha_1 u^2 + \alpha_2 u_x D_x^{-1} u) + \epsilon\,\beta\,u_{xx}$$

and applying the Helmholtz operator

$$H = 1 - \nu\,\epsilon\,\partial^2$$

to achieve:

- coefficients at terms $\mu^2 u_x u^2$ and $\epsilon^2 u_{xxxxx}$ to vanish
- the ratios of the coefficients at the terms $\mu\epsilon u_{xxx}u, \mu\epsilon u_{xx}u_x$ and ϵu_{txx} are (2 : 1 : 1)

Then

$$\nu = \frac{19}{60}, \quad \alpha_1 = \frac{7}{20}, \quad \alpha_2 = -\frac{1}{5}, \quad \beta = \frac{1}{30}.$$

After a re-scaling and a Galilean transformation the euqtion (in orders $1, \epsilon, \mu, \epsilon\mu$ and **ignoring** terms in orders $\epsilon^3, \epsilon^2\mu, \epsilon\mu^2, \mu^3$) can be brought to the "standard Camassa-Holm" form:

$$V_T - V_{TXX} + 2\omega V_X + 3VV_X - 2V_XV_{XX} - VV_{XXX} = 0$$

Where $\omega = \frac{c_0 10\sqrt{285}}{h_0 361}$. In the special case $\omega = 0$ this equation is called "the peakon equation".

The correction to the equation due to the terms of orders $\epsilon^3, \epsilon^2\mu, \epsilon\mu^2, \mu^3$ is: (here $W_X = V$)

$$\frac{12}{361}\frac{V_{X,X}V_XW^2}{\omega^2} + \frac{2440}{361}\frac{V_{X,X}V_XV}{\omega} + \frac{2765}{722}\frac{{V_X}^3}{\omega} + \omega\frac{446}{2527}V_{X,X,X,X,X,X,X}$$
$$+\frac{1806}{361}V_XV_{X,X,X,X} - \frac{24}{361}\frac{WV_{X,X,X,X}V}{\omega} + \frac{24}{361}\frac{WV_{X,X}^2}{\omega} - \frac{6}{361}\frac{W\ V_{X,X}V^2}{\omega^2}$$
$$+\frac{610}{361}V_{X,X}V_{X,X,X} + \frac{632}{361}V_{X,X,X,X,X}V + \frac{340}{361}\frac{V_{X,X,X}V^2}{\omega}$$

Reduction group and automorphic Lie algebras

$$\downarrow$$

Lax representations for integrable PDEs

$$\downarrow$$

Darboux transformations

$$\downarrow$$

Integrable discrete systems

Automorphic Lie Algebras

$\mathbb{C}(\lambda)$ is a field of rational functions of λ
$\mathcal{A}$ is a semisimple finite dimensional Lie algebra over $\mathbb{C}$.

$$\mathcal{A}(\lambda) = \mathcal{A} \otimes \mathbb{C}(\lambda) = \{\sum_k a_k \cdot f_k(\lambda) \,|\, a_k \in \mathcal{A}, f(\lambda) \in \mathbb{C}(\lambda)\}$$

$$a(\lambda) = \sum_k a_k \cdot f_k(\lambda), \quad b(\lambda) = \sum_s b_s \cdot g_s(\lambda)$$

$$[a(\lambda), b(\lambda)] = \sum_{k,s} [a_k, b_s] \cdot f_k(\lambda) g_s(\lambda)$$

$\mathrm{Aut}\,\mathcal{A} \times \mathrm{Aut}\,\mathbb{C}(\lambda)$ is a group of automorphisms of $\mathcal{A}(\lambda)$.

$PSL(2, \mathbb{C})$ is a group of automorphisms of $\mathbb{C}(\lambda)$

$$f(\lambda) \mapsto f(\sigma^{-1}(\lambda)), \quad \sigma(\lambda) = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}, \quad \alpha\delta - \beta\gamma \neq 0$$

Let G be a finite subgroup of $PSL(2, \mathbb{C})$. According F.Klein G is one of the following:

1. the additive group of integers modulo N, $\mathbb{Z}_N$

2. the symmetry group of the dihedron with N vertices, $\mathbb{D}_N$ (special case $\mathbb{D}_2 \sim \mathbb{Z}_2 \times \mathbb{Z}_2$)

3. the symmetry group of the tetrahedron, $\mathbb{T}$

4. the symmetry group of the octahedron, $\mathbb{O}$

5. the symmetry group of the icosahedron, $\mathbb{I}$

Let ρ be a homomorphism $\rho : G \mapsto$ Aut $\mathcal{A} \times$ Aut $\mathbb{C}(\lambda)$, the image of $g \in G$ in Aut $\mathcal{A} \times$ Aut $\mathbb{C}(\lambda)$ is a pair (ρ_g, σ_g), where $\rho_g \in$ Aut $\mathcal{A}$ and $\sigma_g \in PSL(2, \mathbb{C})$.

The group $G_R = \rho(G)$ we call the reduction group (introduced by AVM in 1979).

An invariant subalgebra $\mathcal{A}^{G_R}(\lambda) \subset \mathcal{A}(\lambda)$, where

$$\mathcal{A}^{G_R}(\lambda) = \{a(\lambda) \in \mathcal{A}(\lambda) \,|\, \rho_g(a(\sigma_g^{-1}(\lambda)) = a(\lambda)\}$$

we call the G_R automorphic Lie algebra.

Let $\mu \in \bar{\mathbb{C}}$, we denote $\mathcal{R}_G(\lambda; \mu)$ the ring of rational functions of λ with poles in $G(\mu) = \{\sigma\mu \,|\, \sigma \in G\}$ and no other singularities. G is a group of automorphisms of $\mathcal{R}_G(\lambda; \mu)$.

Example: $\mathbb{Z}_2$ reduction group. Let $\mathcal{A} = sl_2(\mathbb{C}) \sim A_1$ and thus all automorphisms of $\mathcal{A}$ are inner automorphisms

$$a \mapsto Ad_U(a) = UaU^{-1}, \qquad U \in PSL(2,\mathbb{C}).$$

Let the group $G = \mathbb{Z}_2$ is generated by transformation $\kappa_1 : \lambda \mapsto -\lambda$, the reduction group is generated by $g_1 = (Ad_{\sigma_3}, \kappa_1)$, thus the corresponding automorphic Lie algebra is

$$\mathcal{A}^{\mathbb{Z}_2}(\lambda) = \{a(\lambda) \in sl_2 \otimes \mathbb{C}(\lambda) \,|\, \sigma_3 a(-\lambda)\sigma_3 = a(\lambda)\}$$

Its subalgebra (κ_1 is an automorphism of $\mathbb{C}[\lambda] = \mathcal{R}_{\mathbb{Z}_2}(\lambda;\infty)$)

$$sl_2(\lambda;\infty)^{\mathbb{Z}_2} = \{a(\lambda) \in sl_2 \otimes \mathbb{C}[\lambda] \,|\, \sigma_3 a(-\lambda)\sigma_3 = a(\lambda)\} \sim A_1^1$$

is isomorphic to the affine sl_2 algebra.
Moreover $sl_2(\lambda;\infty)^{\mathbb{Z}_N} \sim A_1^1$ for any $N > 1$ (V.Kac).

Example: $\mathbb{D}_2$ reduction group. Let $\mathcal{A} = sl_2(\mathbb{C})$ the reduction group G_R is generated by $g_1 = (Ad_{\sigma_3}, \kappa_1)$ as above and $g_2 = (Ad_{\sigma_1}, \kappa_2)$, where $\kappa_2 : \lambda \mapsto \lambda^{-1}$ ($g_1^2 = g_2^2 = g_1 g_2 g_1 g_2 = id$, thus $G_R \sim \mathbb{Z}_2 \times \mathbb{Z}_2 \sim \mathbb{D}_2$).

$\{a(\lambda) \in sl_2 \otimes \mathbb{C}[\lambda, \lambda^{-1}] \,|\, \sigma_3 a(-\lambda)\sigma_3 = \sigma_1 a(\lambda^{-1})\sigma_1 = a(\lambda)\}$

Algebra $sl_2(\lambda; 0)^{\mathbb{D}_2}$ is generated by

$$\mathbf{a}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix},$$

$$\mathbf{a}_3 = (\lambda^2 - \lambda^{-2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\mathbf{a}_1, \mathbf{a}_3] = 4\mathbf{a}_2 - 2J\mathbf{a}_1, \quad [\mathbf{a}_2, \mathbf{a}_3] = -4\mathbf{a}_1 + 2J\mathbf{a}_2$$

$$[\mathbf{a}_1, \mathbf{a}_2] = \mathbf{a}_3, \quad J = \lambda^2 + \lambda^{-2}.$$

Affine A_1^1 and $sl_2(\lambda;0)^{\mathbb{D}_2}$ are infinite dimensional Lie algebras over $\mathbb{C}$. They are not isomorphic.

A_1^1 and $sl_2(\lambda;0)^{\mathbb{D}_2}$ are 3-dimensional Lie algebras over the fields $\mathbb{C}(\lambda^2)$ and $\mathbb{C}(\lambda^2+\lambda^{-2})$ respectively.

Remarkable result: A_1 automorphic Lie algebras corresponding to the groups $\mathbb{D}_N, \mathbb{T}, \mathbb{O}, \mathbb{I}$ are all isomorphic. Thus it is sufficient to study $\mathbb{D}_2$ reduction group.

In the A_2 case we have checked that the tetrahedron ($\mathbb{T}$) and icosahedron ($\mathbb{I}$) automorphic Lie algebras are isomorphic, but they are not isomorphic to the dihedral ($\mathbb{D}_3$) and affine A_2^1, A_2^2 Lie algebras.

$$L = \partial_x + \sum_{i=1}^{3} u_i(x,t)\mathbf{a}_i$$

$$M = \partial_t + \sum_{i=1}^{3} v_i(x,t)\mathbf{a}_i + \sum_{i=1}^{3} w_i(x,t)J\mathbf{a}_i$$

$$\mathbb{Z}_2 : \quad u_t = -\frac{1}{2}(u^2 v)_x - \frac{1}{2}u_{xx}, \quad v_t = -\frac{1}{2}(uv^2)_x + \frac{1}{2}v_{xx}$$

$$\mathbb{D}_2 : \quad \begin{aligned} u_t &= -\tfrac{1}{2}(u^2 v)_x - \tfrac{1}{2}u_{xx} + 2v_x \\ v_t &= -\tfrac{1}{2}(uv^2)_x + \tfrac{1}{2}v_{xx} + 2u_x \end{aligned}$$

Darboux Transformations and discrete integrable systems.

$$L = D_x + U(x,t;\lambda), \qquad \widehat{L} = D_x + \widehat{U}(x,t;\lambda)$$

$$L\Psi = 0 \longrightarrow \widehat{L}\widehat{\Psi} = 0, \qquad \widehat{\Psi} = M(U,\widehat{U},\lambda)\Psi$$

$$\det(M) = const(\lambda) \neq 0$$

$$D_x(M) + \widehat{U}M - MU = 0$$

$$GL(\sigma^{-1}(\lambda))G^{-1} = L(\lambda) \;\Rightarrow\; GM(\sigma^{-1}(\lambda))G^{-1} = const_1(\lambda)M(\lambda)$$

Example: NLS

$$\mathsf{L} = \mathsf{D}_x + \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

Darboux transformation

$$\mathsf{M} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \mathsf{M}_0$$

$$\det \mathsf{M} = \text{const.} \Longrightarrow \mathsf{M}_0 = \begin{pmatrix} 1 & f \\ g & \alpha + fg \end{pmatrix}, \quad \alpha \in \mathbb{C}$$

$$f = \frac{\tilde{u}}{2}, \quad g = \frac{v}{2}, \quad M = \begin{pmatrix} 1 & \frac{\tilde{u}}{2} \\ \frac{v}{2} & \alpha - \lambda + \frac{\tilde{u}v}{4} \end{pmatrix}$$

$$\tilde{u}_{,x} = 2\,u - \frac{1}{2}\tilde{u}(4\alpha + \tilde{u}v)\,, \quad v_{,x} = \frac{1}{2}\tilde{u}v^2 + 2\,\alpha\,v - 2\,\tilde{v}$$

Another Darboux transformation

$$\mathsf{N} = \begin{pmatrix} \gamma - \lambda + \frac{1}{4}u\widehat{v} & -\frac{1}{2}u \\ -\frac{1}{2}\widehat{v} & 1 \end{pmatrix}$$

Discrete Lax pair

$$\Psi_{n+1,m} = \begin{pmatrix} 1 & u_{n+1,m} \\ v_{n,m} & v_{n,m}u_{n+1,m} + \alpha - \lambda \end{pmatrix} \Psi_{n,m}\,,$$

$$\Psi_{n,m+1} = \begin{pmatrix} 1 & u_{n,m+1} \\ v_{n,m} & v_{n,m}u_{n,m+1} + \beta - \lambda \end{pmatrix} \Psi_{n,m}$$

Discrete system

$$v_{n+1,m} - v_{n,m+1} = \frac{\alpha - \beta}{1 + v_{n,m}u_{n+1,m+1}} v_{n,m}$$

$$u_{n+1,m} - u_{n,m+1} = \frac{\beta - \alpha}{1 + v_{n,m}u_{n+1,m+1}} u_{n+1,m+1}$$

Composition of M and N Darboux transformations

$$\Psi_{n+1,m} = \begin{pmatrix} 1 & u_{n+1,m} \\ v_{n,m} & v_{n,m}u_{n+1,m} + \alpha - \lambda \end{pmatrix} \Psi_{n,m}\,,$$

$$\Psi_{n,m+1} = \begin{pmatrix} \beta - \lambda + u_{n,m}v_{n,m+1} & -u_{n,m} \\ -v_{n,m+1} & 1 \end{pmatrix} \Psi_{n,m}$$

Discrete system

$$v_{n+1,m+1} - v_{n,m} = \frac{\alpha - \beta}{1 + v_{n,m+1}u_{n+1,m}} v_{n,m+1}$$

$$u_{n+1,m+1} - u_{n,m} = \frac{\beta - \alpha}{1 + v_{n,m+1}u_{n+1,m}} u_{n+1,m}$$

Dihedral group $\mathbb{D}_2$

Lax operator

$$\mathsf{L} = \mathsf{D}_x + \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix} - \frac{1}{\lambda^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Darboux transformation

$$\mathsf{M} = \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} + r\,\mathrm{I} + \frac{1}{\lambda} \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$$

Determinant conditions

$$rf = \alpha + p\,q\,, \quad f^2 - p^2 + q^2 = \beta - r^2, \quad \alpha, \beta \in \mathbb{C}$$

Compatibility condition

$$p = \frac{1}{2} f\,u\,, \quad q = \frac{1}{2} f\,\tilde{v}$$

Toda system: Lax operator $N \times N$

$$\mathsf{L} = \mathsf{D}_x + \phi_{,x} + \lambda\,\Delta\,, \quad \phi_{,x} = \mathsf{diag}\,(\phi^i_{,x})\,, \quad \Delta = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

Darboux transformations

$$\mathsf{M}_k = (\mathsf{A}_k + \lambda\,\Delta)\,\Delta^k\,, \quad \mathsf{A}_k = \alpha_k\,\mathsf{diag}\Big(\exp(\tilde{\phi}^i - \phi^{i+k})\Big)\,,$$

$$\alpha_k \in \mathbb{C}^*\,, \quad k = 0, \ldots, N-1$$

Discrete Lax operators ($N = 3$)

$$\Psi_{n+1,m,k} = \Big(\alpha_0\,\mathsf{diag}\Big(\exp(\tilde{\phi}^i - \phi^i)\Big) + \lambda\,\Delta\Big)\,\Psi_{n,m,k}$$

$$\Psi_{n,m+1,k} = \Big(\alpha_1\,\mathsf{diag}\Big(\exp(\tilde{\phi}^i - \phi^{i+1})\Big) + \lambda\,\Delta\Big)\,\Delta\,\Psi_{n,m,k}$$

$$\Psi_{n,m,k+1} = \Big(\alpha_2\,\mathsf{diag}\Big(\exp(\tilde{\phi}^i - \phi^{i+2})\Big) + \lambda\,\Delta\Big)\,\Delta^2\,\Psi_{n,m,k}$$

Discrete systems

$$u^i_{n+1,m+1,k} = \frac{u^i_{n,m+1,k}\, u^j_{n+1,m,k}}{u^\ell_{n,m,k}}\, \frac{a_1\, u^j_{n,m+1,k} - a_0\, u^\ell_{n+1,m,k}}{a_1\, u^i_{n,m+1,k} - a_0\, u^j_{n+1,m,k}}$$

$$u^i_{n+1,m,k+1} = \frac{u^i_{n,m,k+1}\, u^\ell_{n+1,m,k}}{u^i_{n,m,k}}\, \frac{a_2\, u^j_{n,m,k+1} - a_0\, u^i_{n+1,m,k}}{a_2\, u^i_{n,m,k+1} - a_0\, u^\ell_{n+1,m,k}}$$

$$u^i_{n,m+1,k+1} = \frac{u^j_{n,m,k+1}\, u^\ell_{n,m+1,k}}{u^j_{n,m,k}}\, \frac{a_2\, u^\ell_{n,m,k+1} - a_1\, u^j_{n,m+1,k}}{a_2\, u^j_{n,m,k+1} - a_1\, u^\ell_{n,m+1,k}}$$

$$\{i,\, j,\, \ell\} \in \{1, 2, 3\}$$