THE INTERACTION OF NEAR-LIMIT SOLITONS IN STRONGLY NONLINEAR MODELS OF INTERNAL WAVES

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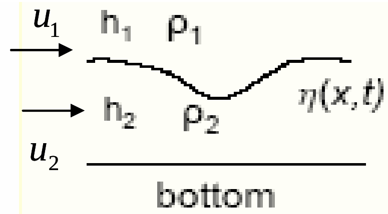
From weakly nonlinear model equations(KdV, Bussinesq, Gardner and ...)

To strongly nonlinear long-wave equations

Choi,Camassa(CC-model),1999

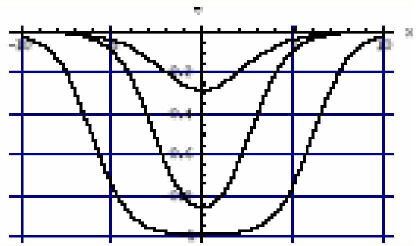
$h_{1,2}$



$$\eta_{1,2t} + \left(\eta_{1,2}\, u_{1,2}\right)_x = 0;$$

$$\rho_{1,2}(u_{1,2t} + u_{1,2}u_{1,2x} + g\xi_x) = -p_x + \frac{1}{3\eta_{1,2}}\left(\eta_{1,2}^3\left(\frac{\partial}{\partial t} + u_{1,2}\frac{\partial}{\partial x}\right)^2 \xi\right)_x$$

$\eta_{1,2} = h_{1,2} \mp \xi$ the perturbed thicknesses of layers



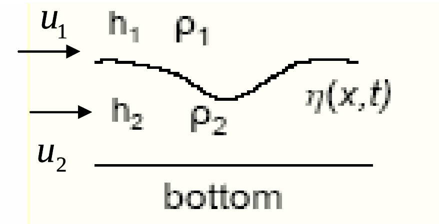
$\xi(x,t)$ the vertical displacement of the liquid layer boundary

$u_{1,2}$ mean (along the vertical coordinate) values of the horizontal velocity of liquid in the layers

$\rho_{1,2}$ the densities of liquid in each layer

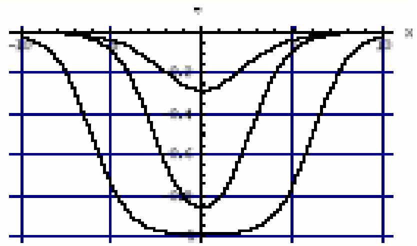
Ostrovsky, Grue (OG),2003

$$\zeta_t + c(\zeta)\zeta_x + \beta(\zeta)\zeta_{xxx} = 0$$



$c(\zeta) -$ velocity of simple(Reman) waves

$$\beta(\zeta) = 1/6c(\zeta)(h_1 + \zeta)(h_2 - \zeta)$$

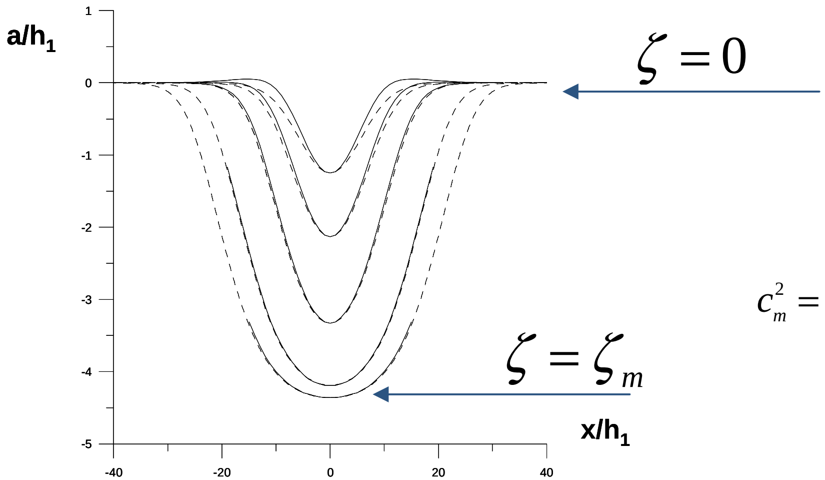


Local dispersion parametercorrespondin to that KdV

Strongly nonlinear solitons and kinks

Family of solitons

$$h_1 / h_2 = 1/10, \rho_1/\rho_2 = 0.997$$



Possible values of solitons amplitudes(a) and velocities(c)

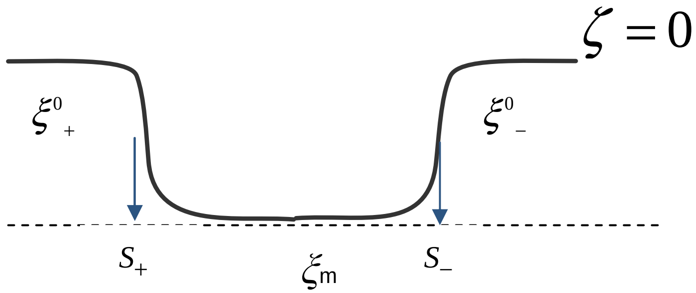
$$c_0^2 < c^2 < c_m^2, \; 0 < |a|^2 < |\xi_m|^2$$

$$c_m^2 = g(h_1 + h_2)\frac{1-(\rho_1/\rho_2)^{1/2}}{1+(\rho_1/\rho_2)^{1/2}} \qquad c_0^2 = gh_1h_2\frac{\rho_2-\rho_1}{\rho_1 h_2 + \rho_2 h_1}$$

$$\xi_m = \frac{h_1 - h_2(\rho_1/\rho_2)^{1/2}}{1+(\rho_1/\rho_2)^{1/2}}$$

$1. a \to 0 (c \to c_0) -$ KdV solitons

$2. a \to \zeta_m (c \to c_m) -$ Strongly nonlinear solitons



$$\xi^0_s(x,t) \approx \xi^0_+(x - c_m t - S_+) + \xi^0_-(x - c_m t - S_-) - \xi_m$$

compaund of kinks of opposite polarities ξ^0_+ ξ^0_-

Behavior of kinks near their asimptotic $\zeta = 0, \quad \xi = \zeta_m$

$$\xi_+ = \begin{cases} e_0 e^{\Lambda_0 (X - S_+)} npu \ (X - S_+) \to -\infty \\ \xi_m + e_m e^{-\Lambda_m (X - S_+)} npu \ (X - S_+) \to +\infty \end{cases}$$

$$\xi_- = \begin{cases} \xi_m + e_m e^{\Lambda_m (X - S_-)} \ npu (X - S_-) \to -\infty \\ e_0 e^{-\Lambda_0 (X - S_-)} \ npu (X - S_-) \to +\infty \end{cases}$$

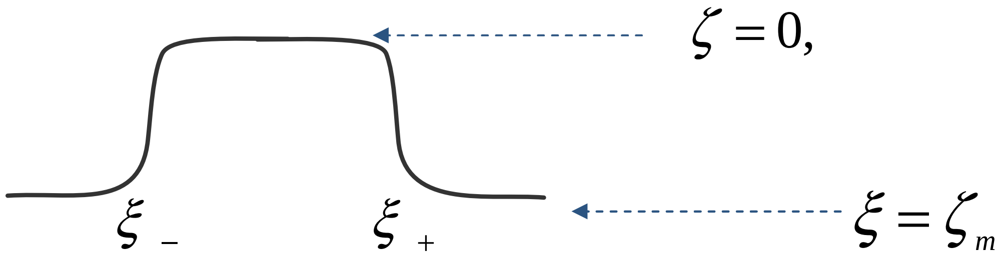
It is important for future:

1.$\Lambda_0 \neq \Lambda_m$ (in general case)

2.compaund of transposing kinks $((S_+ - S_-) > 0)$

is a solitons opposite polarity with non-zero asymptotic: $\zeta_s \to \zeta_m$

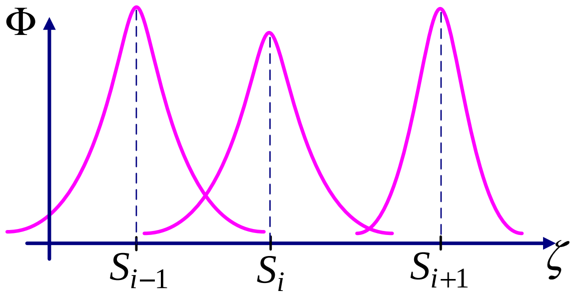
$$\xi^0{}_s(x,t) \approx \xi^0{}_+ (x - c_m t - S_+) + \xi^0{}_- (x - c_m t - S_-))$$



Interaction of solitons close to limiting (equations of motion for kinks)

1.CC and OG – nonintegrable systems 2.Numerical simulation does not give a qualitative understanding

3.As a possibility-approximate description of solitons interaction as classical particles:



Field structure:

$$\zeta(x,t) = \sum_i^N \zeta_s(x - c\,t - S_i(t), c_m + \dot{S}_i)$$

$$+\sum \varepsilon^n \zeta^n(x,t), \varepsilon \square \dot{S}_i / c$$

Equation for coordinates of solitons:

$$m d^2 S_i / dt^2 = \alpha[e^{-\Lambda_0(S_i - S_{i-1})} - e^{-\Lambda_0(S_i - S_{i+1})}]$$

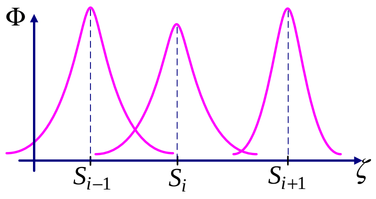
$m = \partial P / \partial c$ "mass" soliton

$P -$ *full* momentum of soliton

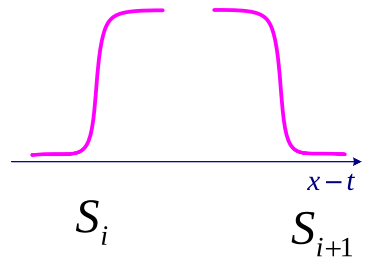
$\alpha e^{-\Lambda_0(S_i - S_{i-1})} -$ product exponential asymptotics of neighboring solitons

For limiting solitons: width,momentum → ∞

Description of transformation length,momentum of solitons- become ineffective



Solitons interaction → Interaction of kinks



Structure of global solution

$$\zeta\ (x,t)=\sum_{i=1}^{2N}\zeta_i^{(0)}\left(x-c_m t-S_i(t)\right)+\sum\varepsilon^n\zeta^n-N\zeta_m$$

$$\frac{dS_i}{dt}=-\begin{cases}M_0e^{-\Lambda_0(S_i-S_{i-1})}+M_me^{-\Lambda_m(S_{i+1}-S_i)}, & i=2p+1\\ M_me^{-\Lambda_m(S_i-S_{i-1})}+M_0e^{-\Lambda_0(S_{i+1}-S_i)}, & i=2p\end{cases}$$

substitution

$$R_i=\begin{cases}\Lambda_m(S_{i+1}-S_i)-\ln(\Lambda_0M_m), & i=2p-1\\ \Lambda_0(S_{i+1}-S_i)-\ln(\Lambda_mM_0), & i=2p\end{cases}$$

$$\frac{dR_i}{dt}=e^{-R_{i-1}}-e^{-R_{i+1}}$$

Integrable system Kac-Mierbeke

Π-flux momentum

Conservation law: $\partial P/\partial t+\partial\Pi/\partial x=0$

For solitons:

$$P\to P+\Delta P,\Delta P\sim\dot{S}\partial P/\partial c\to\partial P/\partial t\sim\ddot{S}$$

For kinks: $P\to P+\Delta P,\Delta P\sim\Delta S\to\partial P/\partial t\sim\dot{S}$

Explicit form of general solution

General solution can be obtain by using the solution for kinks coordinates found earlier for Gardner model

$$s_{i=2p-1} = \ln[\frac{a^{(+)}_{N-p+1}}{a^{(+)}_{N-p}}.]; s_{\substack{i=2p \\ (p=2,3,..)}} = \ln[\frac{a^{(-)}_{N-p+1}}{a^{(-)}_{N-p}}]$$

$$a^{(\pm)}_{N} = 1, a^{(\pm)}_{N-1} = \sum_{j=1}^{N} \exp[\varepsilon_j (t - T_j) \pm \delta_j]$$

$$a^{(\pm)}_{N-2}(t) = \sum_{1<j_1<j_2}^{N} \exp[\varepsilon_{j_1}(t - T_{j1}) + \varepsilon_{j_2}(t - T_{j_2}) \pm \delta_{j_1} \pm \delta_{j_2} \pm A_{j_1,}]$$

$$a^{(\pm)}_{N-p}(t) = \sum_{1<j_1<j_2 \dots j_p}^{N} \exp[\sum_{k=1}^{p} [\varepsilon_{j_k}(t - T_{j_k}) \pm \delta_{j_k}] + \sum_{1<j_k<j_l}^{p} A_{jk,jl}],$$

$$a^{(\pm)}_{0} = \exp[\sum_{j=1}^{N} \exp[\varepsilon_j (t - T_j) \pm \delta_j] + \sum_{1<j<j_l}^{N} A_{j,j1}],$$

The differences $r_i = s_{i+1} - s_i$, in it satisfy the same equations as Ri in the present case (CC-model)

where ε_j, T_j are independent parameters characterizing relative velocities and positions of solitons

$$\delta_j = (\ln(2/\varepsilon_j))/2, \exp A_{j_k j_l} = (\varepsilon_{jk} - \varepsilon_{j_l})^2/4$$

The total phase shift of the kinks centers

$$\Delta S_p = \frac{1}{\Lambda_0}\left(\sum_{q=1}^{p-1} A_{pq} - \sum_{q=p+1}^{N} A_{pq}\right) - (\Lambda_m{}^{-1} - \Lambda_0{}^{-1})\left(\sum_{q=1}^{p-1}\delta_q - \sum_{q=p+1}^{N}\delta_q\right)$$
$$+(N-1)[\Lambda_m{}^{-1}\ln\left(\Lambda_0 M_m\right) + \Lambda_0{}^{-1}\ln\left(\Lambda_m M_0\right)]$$

The total phase shift ΔS_p is equal to the

sum of particle phase shifts $\Delta S_{p,q}$ → are due to the collisions of a given soliton separetely with each of the other

$$\Delta S_{p,q} = \pm\frac{1}{\Lambda_0}A_{p,q} \pm \left(\frac{1}{\Lambda_m} - \frac{1}{\Lambda_0}\right)\delta_q + \frac{1}{\Lambda_m}\ln(M_m\Lambda_0)$$

$\pm$ correspond to the cases of p>q and p<q

For solitons of Gardner equation

$$\Lambda = \Lambda_0, M_0 = M_m$$

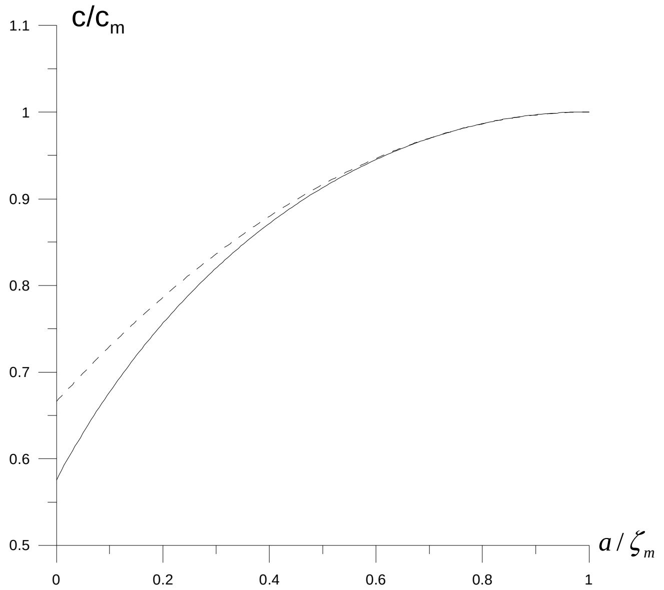
$$\Delta S_{p,q} = \pm A_{p,q} = \pm\ln\left(\frac{\varepsilon_p - \varepsilon_q}{2}\right)^2$$

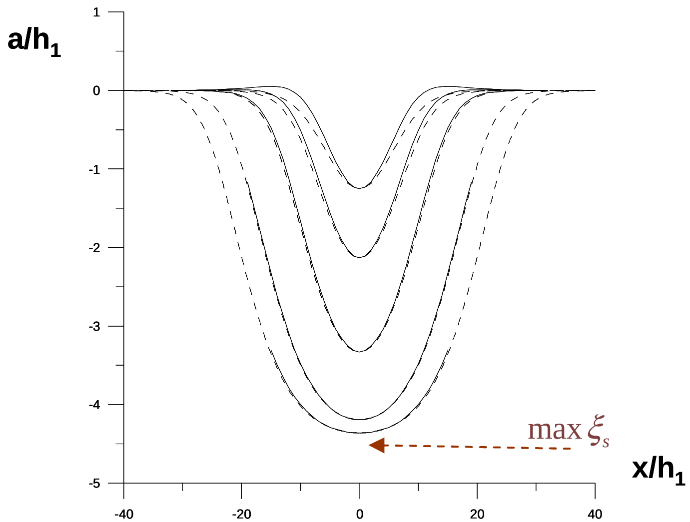
Compaund solitons and their interactions

A singl soliton(two-kink solution)

$$\xi_s(x-ct)=\xi_1(X-S_1)+\xi_2(X-S_2)-\xi_m+0(\varepsilon)$$

$$h_1/h_2=1/10, \rho_1/\rho_2=0.997$$





$$c_m-c=\Lambda^{-1}{}_0\varepsilon == M_m e^{-\Lambda_m(S_2-S_1)} \quad \Longleftrightarrow \quad a=\max\xi_s == \xi_m + 2e_m e^{-\Lambda_m(S_1-S_2)/2}$$

$$S_1=-\Lambda_0^{-1}\varepsilon t \ , S_2=-\Lambda_0^{-1}\varepsilon t+\Lambda_m^{-1}\left(2\delta+\ln M_m\Lambda_0\right)$$

Interaction of two solitons

$$\zeta(x,t)=\sum_{i=1}^{4}\zeta_i^{(0)}(X-S_i)-2\zeta_m+0(\varepsilon),$$

$$S_1=-\Lambda_0^{-1}\cdot\ln a_1^{(+)} \qquad S_2=-\Lambda_0^{-1}\cdot\ln a_1^{(+)}+\Lambda_m^{-1}\cdot\ln\frac{a_1^{(+)}}{a_1^{(-)}}+\Lambda_m^{-1}\cdot\ln(M_m\Lambda_0)$$

$$S_3=\Lambda_0^{-1}\cdot\ln\frac{a_1^{(-)}}{a_1^{(+)}}+\Lambda_m^{-1}\cdot\ln\frac{a_1^{(+)}}{a_1^{(-)}}+\Lambda_m^{-1}\cdot\ln(M_m\Lambda_0)+\Lambda_0^{-1}(M_0\Lambda_m)$$

$$S_4=\Lambda_0^{-1}\cdot\ln\frac{a_1^{(-)}}{a_1^{(+)}}+2\Lambda_m^{-1}\cdot\ln(M_m\Lambda_0)+\Lambda_0^{-1}\cdot\ln(M_0\Lambda_m)+2\Lambda_m^{-1}(\delta_1+\delta_2)$$

where ε_j, T_j are independent parameters characterizing relative velocities and positions of solitons

$$a_1^{(+)}=e^{\varepsilon_1(t-T_1)\pm\delta_1}+e^{\varepsilon_2(t-T_2)\pm\delta_2},$$

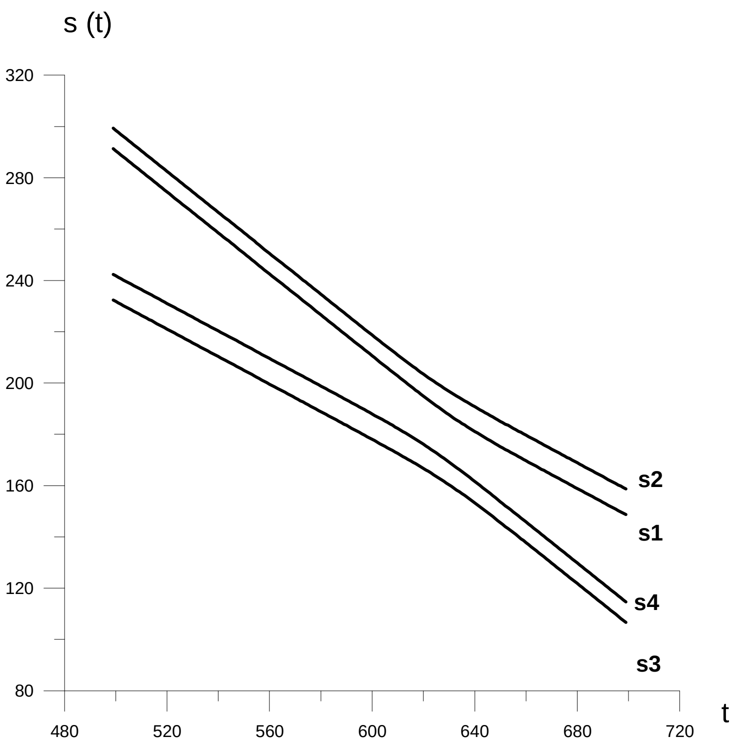
$$a_0^{(+)}=\exp[\varepsilon_1(t-T_1)+\delta_1+\varepsilon_2(t-T_2)\pm\delta_2+A_{12}]$$

$$\delta_{1,2}=\frac{1}{2}\ln\frac{2}{\varepsilon_{1,2}} \quad A_{12}=\ln\left(\frac{\varepsilon_1-\varepsilon_2}{2}\right)^2,\ \varepsilon_1<\varepsilon_2$$

Two limiting case

$$1. \varepsilon_2 - \varepsilon_1 << \varepsilon_{1,2} -$$

Interaction of soliton as classical particles



Phase shift

$$\Delta s_{1,2} = \pm 1/\Lambda_0 \ln\left(\frac{\varepsilon_1 - \varepsilon_2}{2}\right)^2$$

$$\pm 2\left(\frac{1}{\Lambda_m} - \frac{1}{\Lambda_0}\right) \ln \frac{\overline{2}}{\varepsilon_{1,2}} \approx \pm 1/\Lambda_0 \ln\left(\frac{\varepsilon_1 - \varepsilon_2}{2}\right)^2$$

As for classical particles!

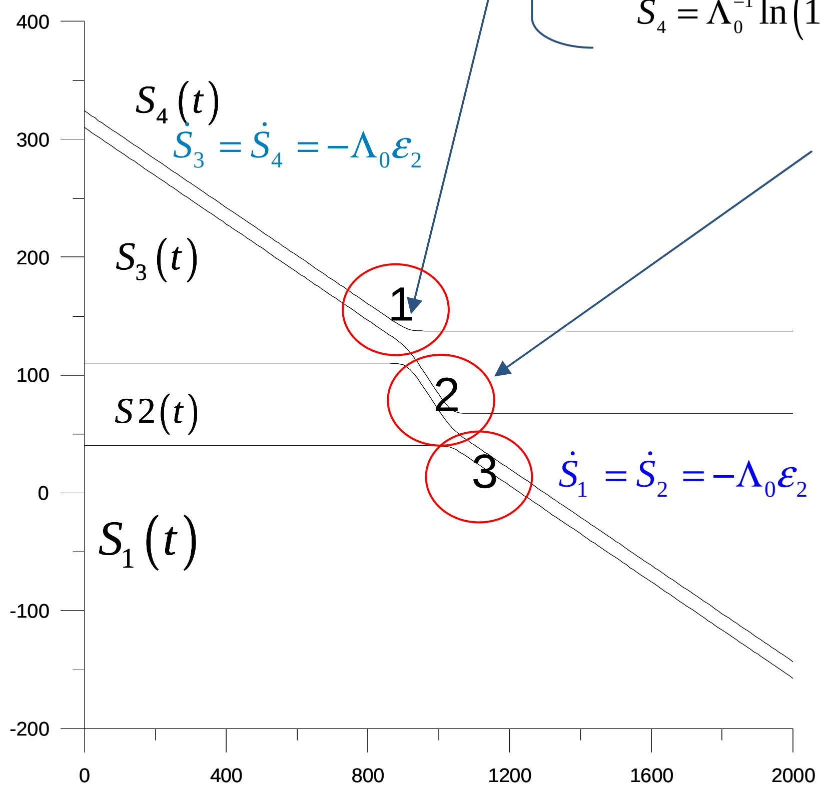
$2.\varepsilon_1$ □ ε_2 □ 1

Three kink interaction $(\varepsilon_1 \to 0, \varepsilon_1 T_1 = \delta_1)$

$$\begin{cases} S_2 = -\Lambda_m^{-1} \ln\left(1 + e^{\varepsilon_2 (t - T_2) - \delta_2}\right) \\ S_3 = -\Lambda_m^{-1} \ln\left(1 + e^{\varepsilon_2 (t - T_2) - \delta_2}\right) + \Lambda_0^{-1} \ln\left(1 + e^{-\varepsilon_2 (t - T_2) + \delta_2}\right) + \Lambda_0^{-1} \ln\left(2 M_0 \Lambda_m / \varepsilon_2\right) \\ S_4 = \Lambda_0^{-1} \ln\left(1 + e^{-\varepsilon_2 (t - T_2) + \delta_2}\right) + \Lambda_0^{-1} \ln\left(2 M_0 \Lambda_m / \varepsilon_2\right) + \Lambda_m^{-1} \ln\left(2 M_m \Lambda_0 / \varepsilon_2\right) \end{cases}$$

Soliton opposite polarity

$\dot{S}_2 = \dot{S}_3 = -\Lambda_m \varepsilon_2$



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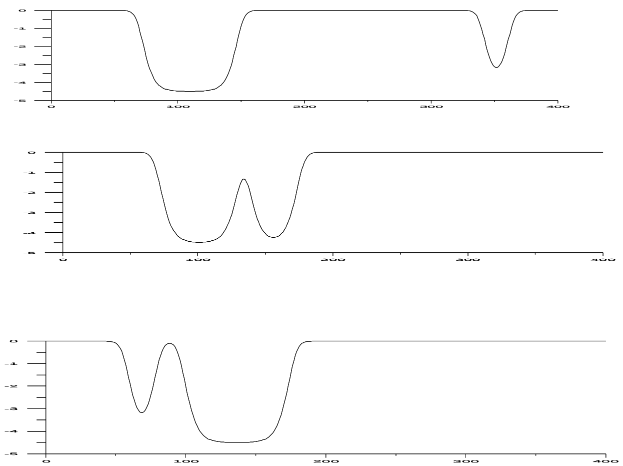
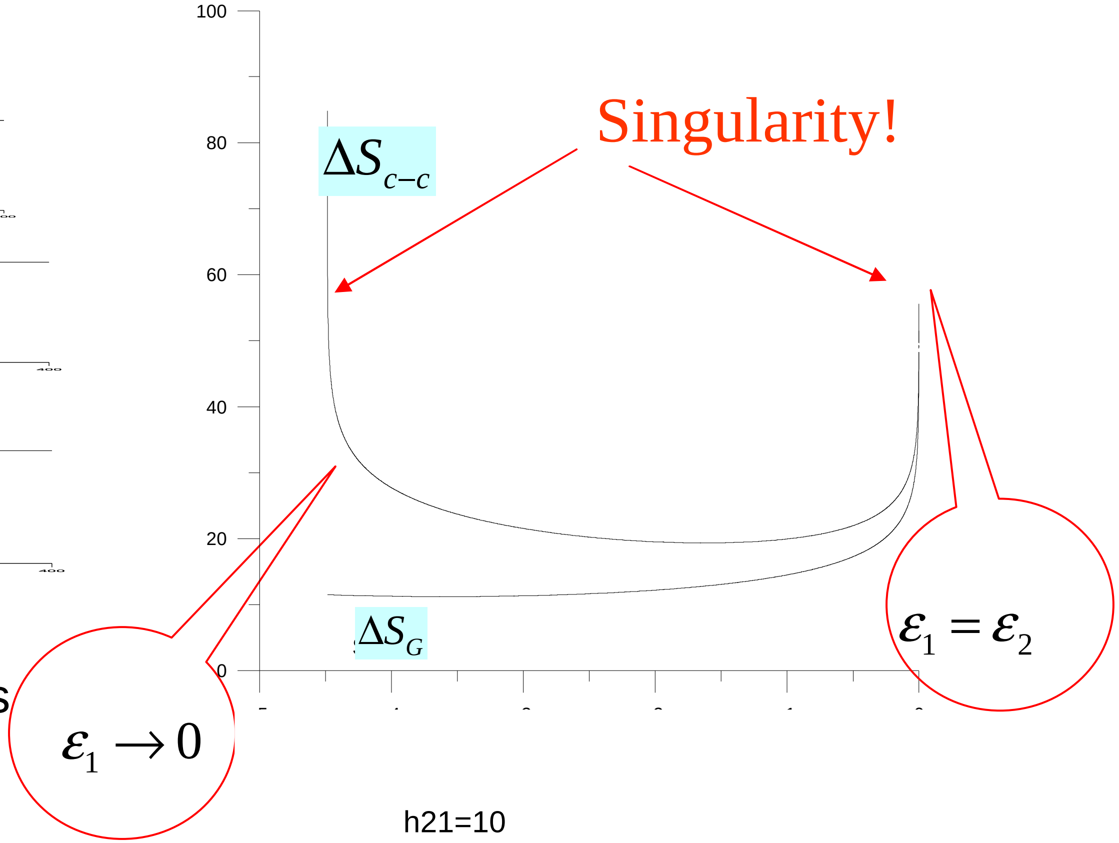
Some features of strong soliton interaction.

$h_1 / h_2 = 1/10, \rho_1 / \rho_2 = 0.997$



Interaction of two solitons



h21=10

The dependencies of phase shifts on ε_1

Concluding remarks

- The description of solitons as compounds of kinks seems to be most adequate to strongly nonlinear waves having a property of approach a limiting, tabletop stage

- **We hope that the theory developed here illustrates the efficiency of perturbation approach for non – integrable equations such as the CC-system**

- **The limiting of a soliton amplitude is a realistic situation in oceanography**