Wave Turbulence in Superfluid $^4$He:
Energy Cascades, Rogue Waves & Kinetic Phenomena in the Laboratory

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Outline

1 Introduction
   • Motivation
2 Modelling wave turbulence
   • Need for models
   • Second sound in He II
3 Experiments & results
   • Experimental set-up
   • Energy cascades
   • Transients & kinetics
4 Discussion
   • Wider implications
   • Conclusion

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Wave Turbulence in Superfluid $^4$He
Cascades of energy through different length scales.
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In addition to familiar vortex turbulence in fluids, turbulence can also occur in systems of waves, e.g. –

- Magnetic turbulence in interstellar gases.
- Shock waves in the solar wind.
- Sound waves in oceanic waveguides.
- Capillary waves on ocean surface.
- Phonon turbulence in solids.
- Second sound in He II...
Wave turbulence arises in systems of strongly interacting nonlinear waves.

It is similar to vortex turbulence in fluids in that, usually –

- There is a flow of energy across the length scales – conventionally, from the scale of the driving towards smaller and smaller scales.
- At small enough scales dissipation (due to e.g. viscosity) becomes important and terminates the cascade.

Dyachenko and Zakharov suggested that rogue waves on the ocean arise through nonlinear wave interactions...
Need for models

- In practice, it is difficult to test the theory of wave turbulence through studies of natural events (e.g. on ocean, in interstellar media).
- So a laboratory test-bed is needed, where parameters can be controlled and adjusted.
- It turns out that the properties of He II make it an ideal model system.
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Sound modes in He II

Two sound modes in bulk He II –

- **First sound** is a pressure-density wave, with in-phase motion of the normal and superfluid components, and phase velocity

\[
U_1 = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_\sigma}
\]
Sound modes in He II

Two sound modes in bulk He II –

- **First sound** is a pressure-density wave, with in-phase motion of the normal and superfluid components, and phase velocity

\[ u_1 = \sqrt{\left( \frac{\partial P}{\partial \rho} \right)_\sigma} \]

- **Second sound** is an entropy-temperature wave, with anti-phase motion of the two components, and phase velocity

\[ u_2 = \sqrt{\frac{\rho_s \sigma^2}{\rho_n} \left( \frac{\partial T}{\partial \sigma} \right)_\rho} \]
For finite temperature excursions \( \delta T \), 2nd sound sound velocity is

\[
u_2 = u_{20}(1 + \alpha \delta T)
\]

where the nonlinear coefficient

\[
\alpha = \frac{\partial}{\partial T} \ln \left( u_{20}^3 \frac{C}{T} \right)
\]

Datapoints: experiments, Dessler & Fairbank (1956).
Curve: theory, I M Khalatnikov (1952)
Nonlinear coefficient for second sound

- For finite temperature excursions $\delta T$, 2nd sound sound velocity is:
  \[ u_2 = u_{20}(1 + \alpha \delta T) \]

  where the nonlinear coefficient
  \[ \alpha = \frac{\partial}{\partial T} \ln \left( u_{20}^3 \frac{C}{T} \right) \]

  Note –
  - $\alpha \to -\infty$ as $T \to T_\lambda$
  - $\alpha$ changes sign at $T = 1.88$ K

Datapoints: experiments, Dessler & Fairbank (1956).
Curve: theory, I M Khalatnikov (1952)
Advantages of second sound for modelling

So second sound offers many advantages as a model system for studying wave turbulence –

- Nonlinear coefficient $\alpha$ can be made very large.
- Also, $\alpha$ can be “tuned” by adjustment of $T$ to be either
  - Positive, or
  - Negative, or
  - Zero.

- The small velocity ($20 \text{ m s}^{-1}$) gives good time resolution and convenient experimental dimensions.
- It is easy to apply a variety of different signals to control the second sound generator.
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Create 2nd sound standing wave with heater.
Detect it with a bolometer.
Sinewave of $\omega$ on heater $\Rightarrow$ 2nd sound at $2\omega$ in He II.
But any waveform can be applied.
Construction of cell

Aspect ratio of actual cell was different –

- Length of quartz spacer – 70 mm
- Inner diameter – 15 mm
- Endplates parallel to better than 1:10^4
- Thin film heater
- Thin film Sn-Cu bolometer
- Bolometer sensitivity – 2.6 V K^{-1}
- \( Q \sim 1000 – 3000 \)
Data recording

Experimental procedure –

- Drive heater from sinewave generator (0.1–100 kHz).
- Second sound wave amplitude $\delta T \sim 0.05–5.0$ mK.
- Corresponding Mach number

$$M = \alpha \delta T \sim 10^{-4} - 10^{-2}.$$ 

- And acoustic Reynolds number

$$Re = \frac{\alpha u_{20} (\partial \delta T / \partial x)}{\gamma \omega} \sim \alpha Q \delta T$$

can be adjusted in range $\sim 1–100$.

- Record time series from bolometer (up to $10^6$ points) and use FFT to compute power spectrum.
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Power spectra of 2nd sound standing waves

- Driving on 31\textsuperscript{st} resonance, \( f_d = 3130 \text{ Hz} \).
- Heat flux \( W \) was –
  - (a) 5.5 mW cm\(^{-2}\)
  - (b) 22 mW cm\(^{-2}\)
- Dashed-line in (b) is \( A_f \propto f^{3/2} \).
- Inset: amplitude at driving frequency v. heat flux.
- Arrows show viscous cut-off frequency.
- Kolmogorov-like direct energy cascade in (b).
Numerical theory

- Calculations for 4 different driving force amplitudes $F_d$ –
  - $\triangle$ 0.01
  - $\Diamond$ 0.05
  - $\bigcirc$ 0.1
  - $\square$ 0.3

- Dashed line is $A_f \propto f^{-1}$.

- Inset: standing wave amplitude at driving frequency for linear (dashed) and nonlinear (full curve) waves.

Very similar to experiments!
Probability densities of higher harmonics

- PDFs after sequential removal of harmonics:
  - $0 \Rightarrow$ Original signal
  - $1 \Rightarrow \geq 1$st harmonic retained
  - $n \Rightarrow > n$th harmonic retained

- Higher harmonics approach Gaussian distribution.
Instability against subharmonic generation

- $T = 2.08$ K (negative nonlinearity), $W = 10$ mW cm$^{-2}$.
- Driving near 96th resonance: 9530.8 Hz (top); 9532.4 (middle); 9535.2 (bottom).
- Arrows –
  - Green: driving frequency
  - Blue: first harmonic
  - Red: region of subharmonic generation
Onset of instability

Energy $E_{LF}$ at low frequencies as a function of ac drive $W$. 

![Graph showing the relationship between energy $E_{LF}$ at low frequencies and ac drive $W$.]
Onset of instability

- Energy $E_{LF}$ at low frequencies as a function of ac drive $W$.
- Onset of low frequency energy generation is of a critical character.

![Graph showing energy $E_{LF}$ as a function of $W$](image-url)
Onset of instability

- Energy $E_{LF}$ at low frequencies as a function of ac drive $W$.
- Onset of low frequency energy generation is of a critical character.
- Onset of an inverse energy cascade...

![Graph showing the relationship between energy $E_{LF}$ and ac drive $W$.](image)
Energy can flow both ways

- Under the right conditions, energy in a turbulent acoustic system can flow towards the low frequency spectral domain.
- Inverse energy cascades are also known in 2-D fluid flows.
- Onset of the inverse cascade sometimes involves metastability and hysteresis.
Conditions for onset instability and inverse cascade

Calculated heat flux $W$ for onset of instability (full line) compared with experiment (data points) for different dimensionless detunings

$$\Delta = \frac{\omega_d - \omega_n}{\omega_n}$$

Bars indicate hysteretic width.
Conditions for onset instability and inverse cascade

Calculated heat flux $W$ for onset of instability (full line) compared with experiment (data points) for different dimensionless detunings

\[ \Delta = \frac{\omega_d - \omega_n}{\omega_n} \]

Bars indicate hysteretic width.

Bifurcation diagram (inset)

- Yellow $\Rightarrow$ instability
- White $\Rightarrow$ stability
- Orange line, soft instability
- Blue lines, hard instability
- Green points, critical points
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A cornucopia of interesting transient effects, e.g.
Isolated “rogue waves”
Complex evolution of spectra...
Direct cascade (blue) appears immediately.

Inverse cascade (orange) develops slowly, taking energy from direct cascade.

Inset: evolution of energy density at low (blue) and high (orange) frequencies.
Switch on drive at $t = 0$.

Near 96\textsuperscript{th} resonance, $W = 42 \text{ mW cm}^{-2}$, $T = 2.08 \text{ K}$.

Direct cascade appears first.

Inverse builds up \textit{slowly}.

Ultimately, nearly continuous.
Turbulent decay is interesting too...

- Turbulent decay after switch-off.
- Harmonics sometimes exhibit oscillatory decay.
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- Turbulent decay after switch-off.
- Harmonics sometimes exhibit oscillatory decay.
- Perhaps due to exchange of energy between harmonics, or with radial modes?
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Origins of rogue waves?

- Dyachenko and Zakharov suggest “modulation instability of Stokes wave ⇒ freak wave” (*JETP Lett*, 2005).
  - First experimental observation of giant low-frequency waves.
  - NB oceanic surface involves 4-wave interactions, not 3-wave as here – but essential physics is similar.
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  - First experimental observation of giant low-frequency waves.
  - NB oceanic surface involves 4-wave interactions, not 3-wave as here – but essential physics is similar.
- Applications to other wave turbulent systems...
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Conclusions

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- An inverse cascade can co-exist with the direct cascade, carrying energy towards frequencies lower than that at which it is pumped into the system.
- It leads to a substantial increase in wave amplitude at low frequency, together with isolated rogue waves.
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- Nonlinear second sound waves exhibit turbulence – with a Kolmogorov-like energy cascade towards high frequencies.
- An inverse cascade can co-exist with the direct cascade, carrying energy towards frequencies lower than that at which it is pumped into the system.
- It leads to a substantial increase in wave amplitude at low frequency, together with isolated rogue waves.
- A huge range of interesting kinetic phenomena remains to be explored...
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