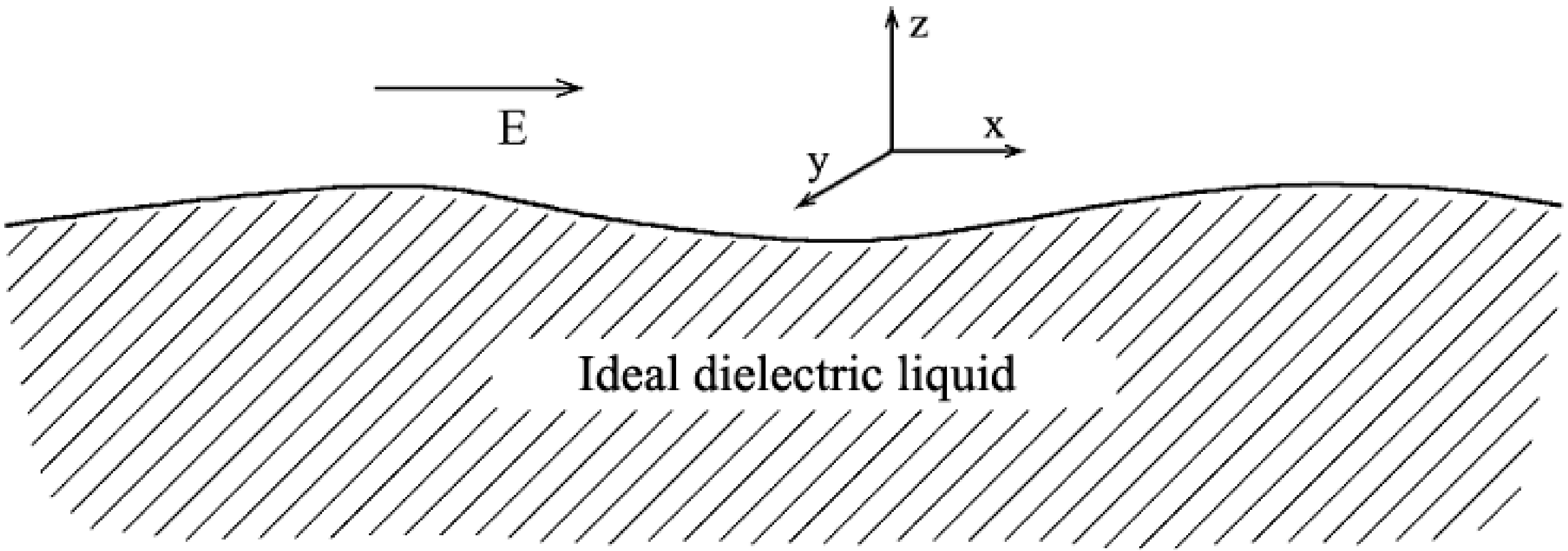
Exact Solutions for Nonlinear Waves on the Surface of a Dielectric Liquid in a Tangential Electric Field in 3D Geometry

N.M. Zubarev and O.V. Zubareva

Institute of Electrophysics, UD, RAS, Ekaterinburg, Russia

The free surface: $z = \eta(x, y, t)$



Dispersion relation for the electrocapillary waves:

$$\omega^2 = \frac{E^2(\varepsilon-1)^2}{4\pi\rho(\varepsilon+1)} k_x^2 + \frac{\alpha}{\rho} k^3, \qquad k^2 = k_x^2 + k_y^2,$$

where
α is the surface tension,
ρ is the density,
ε is the permittivity of the liquid.

If $k \ll \alpha^{-1}(\varepsilon+1)^{-1}(\varepsilon-1)^2 E^2,$

the surface waves propagate without dispersion:

$$\omega^2 = c^2 k_x^2, \qquad c = E(\varepsilon-1)\Big/\sqrt{4\pi\rho(\varepsilon+1)}.$$

Initial equations:

$$\nabla^2\phi = 0, \qquad \nabla^2\varphi = 0, \qquad \nabla^2\varphi' = 0,$$

$$\phi_t + \frac{(\nabla\phi)^2}{2} = \frac{P_E - P_0}{\rho} \equiv \frac{(\varepsilon - 1)(\nabla\varphi \cdot \nabla\varphi' - E^2)}{8\pi\rho}, \qquad z = \eta(x, y, t),$$

$$\eta_t = \phi_z - \nabla_\perp \eta \cdot \nabla_\perp \phi, \qquad z = \eta(x, y, t),$$

$$\varphi = \varphi', \quad \varepsilon \partial_n \varphi = \partial_n \varphi', \qquad z = \eta(x, y, t),$$

$$\phi \to 0, \quad \varphi \to -Ex, \qquad z \to -\infty,$$

$$\varphi' \to -Ex, \qquad z \to \infty,$$

where ϕ, φ, φ' are the velocity and electric potentials,

∂_n denotes the derivative along the normal to the surface.

Limiting case of high permittivity

We consider the case where $\varepsilon \gg 1.$

In this limit we have: $\partial_n \varphi \approx 0, \quad z = \eta(x,y,t),$

i.e., the electric field lines are directed along the tangent to the curved surface.

We find for the electrostatic pressure:

$$P_E \approx \frac{\varepsilon\,(\nabla\varphi)^2}{8\pi}, \quad P_0 \approx \frac{\varepsilon\,E^2}{8\pi} \qquad z = \eta(x,y,t).$$

As a consequence, the surface evolution is defined by the electric field in the liquid.

Water: $\varepsilon \approx 81,$ nitrobenzene: $\varepsilon \approx 36,$ ethyl alcohol: $\varepsilon \approx 26.$

1. Equations of motion

Let us pass to dimensionless variables:

$$\phi \to \lambda E \varepsilon^{1/2}(4\pi\rho)^{-1/2}\phi, \qquad t \to \lambda E^{-1}\varepsilon^{-1/2}(4\pi\rho)^{1/2}t,$$

$$\varphi \to \lambda E\varphi, \qquad x \to \lambda x, \qquad y \to \lambda y, \qquad z \to \lambda z,$$

where λ is the characteristic wavelength. We obtain:

$$\nabla^2\phi = 0, \qquad \nabla^2\varphi = 0,$$

$$\phi_t + \frac{(\nabla\phi)^2}{2} = \frac{(\nabla\varphi)^2}{2} - \frac{1}{2}, \qquad z = \eta(x, y, t),$$

$$\eta_t = \phi_z - \nabla_\perp\eta \cdot \nabla_\perp\phi, \qquad z = \eta(x, y, t),$$

$$\varphi_z - \nabla_\perp\eta \cdot \nabla_\perp\varphi, \qquad z = \eta(x, y, t),$$

$$\phi \to 0, \quad \varphi \to -x, \qquad z \to -\infty.$$

2. Equations of motion

We introduce the auxiliary potential $\tilde{\varphi} \equiv \varphi + x.$

The equations of motion take the following symmetric form:

$$\nabla^2 \phi = 0, \qquad \nabla^2 \tilde{\varphi} = 0,$$

$$\phi_t = -\tilde{\varphi}_x + \frac{(\nabla \tilde{\varphi})^2 - (\nabla \phi)^2}{2}, \qquad z = \eta(x, y, t),$$

$$\eta_t = \phi_z - \nabla_\perp \eta \cdot \nabla_\perp \phi, \qquad z = \eta(x, y, t),$$

$$-\eta_x = \tilde{\varphi}_z - \nabla_\perp \eta \cdot \nabla_\perp \tilde{\varphi}, \qquad z = \eta(x, y, t),$$

$$\phi \to 0, \quad \tilde{\varphi} \to 0, \qquad z \to -\infty.$$

Particular exact solutions

Let $\eta = f(x \mp t, y), \qquad \tilde{\varphi} = \pm\phi = F(x \mp t, y, z).$

Substituting these expressions into the equation of motion, we find:

$$\nabla^2 F = 0,$$

$$-f_x = F_z - \nabla_\perp f \cdot \nabla_\perp F, \qquad z = f,$$

$$F \to 0, \qquad z \to -\infty.$$

This implies that f is an arbitrary function, which uniquely determines the potential F.

So, we have two particular solutions:

$$\eta = f(x - t, y),$$

$$\eta = g(x + t, y).$$

In the linear approximation the evolution of the surface is described by the wave equation:

$$\eta_{tt} = \eta_{xx}, \qquad \eta = f(x - t, y) + g(x + t, y).$$

Small-angle approximation

Let us consider that $|\nabla_\perp \eta| \sim \alpha \ll 1.$

The equations of motion:

$$\psi_t - \hat{k}^{-1}\eta_{xx} = \frac{1}{2}(\hat{k}\psi)^2 - \frac{1}{2}(\nabla_\perp \psi)^2 + \frac{1}{2}\eta_x^2 + \eta\eta_{xx} + \frac{1}{2}(\nabla_\perp \hat{k}^{-1}\eta_x)^2 -$$
$$- \hat{k}^{-1}\partial_x(\eta\hat{k}\eta_x - \nabla_\perp \eta \cdot \nabla_\perp \hat{k}^{-1}\eta_x) + O(\alpha^3),$$

$$\eta_t - \hat{k}\psi = -\hat{k}(\eta\hat{k}\psi) - \nabla_\perp(\eta\nabla_\perp \psi) + O(\alpha^3),$$

where $\psi(x,y,t) \equiv \phi|_{z=\eta},$

$$\hat{k}f = -\frac{1}{2\pi}\iint \frac{f(x',y')}{\left[(x'-x)^2 - (y'-y)\right]^{3/2}} dx'dy'.$$

Interaction of counter-propagating waves

The equation for the surface evolution:

$$\eta_{tt}-\eta_{xx}=\frac{1}{2}\hat{k}\left(\eta_x^2-\eta_t^2+(\nabla_\perp\hat{k}^{-1}\eta_x)^2-(\nabla_\perp\hat{k}^{-1}\eta_t)^2\right)+$$

$$+\nabla_\perp(\eta_x\nabla_\perp\hat{k}^{-1}\eta_x-\eta_t\nabla_\perp\hat{k}^{-1}\eta_t)+O(\alpha^3).$$

The particular exact solutions:

$$\eta(x,y,t)=f(x-t,y),$$

$$\eta(x,y,t)=g(x-t,y).$$

For small but finite α the surface evolution is described by the nonlinear superposition of the oppositely directed waves:

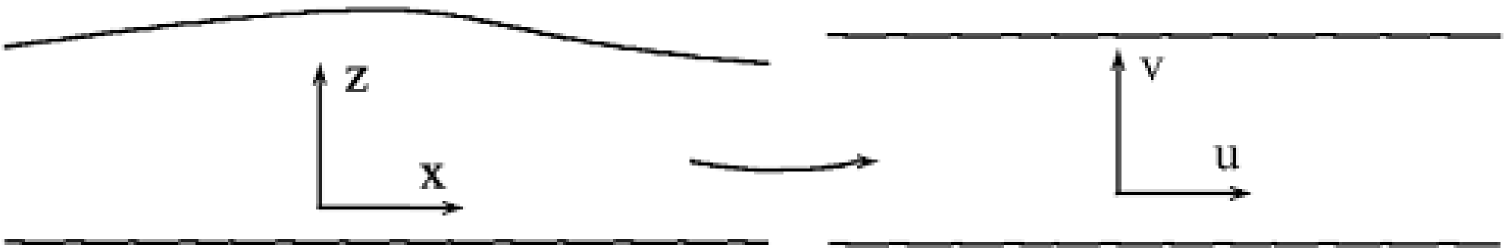
$$\eta(x,y,t)=f(x-t,y)+g(x+t,y)-\frac{1}{2}\hat{k}(fg+\nabla_\perp\hat{k}^{-1}f\cdot\nabla_\perp\hat{k}^{-1}g)-$$

$$-\frac{1}{2}\nabla_\perp(f\nabla_\perp\hat{k}^{-1}g+g\nabla_\perp\hat{k}^{-1}f)+O(\alpha^3).$$

2D geometry; conformal variables

By analogy with the works [A.I. Dyachenko, V.E. Zakharov, E.A. Kuznetsov, Plasma Phys. Rep., 1996; A.I. Dyachenko, E.A. Kuznetsov, M.D. Spector, V.E. Zakharov, Phys. Lett. A, 1996], we perform a conformal mapping of the region occupied by the liquid into a half-plane:

$$-\infty < v < 0, \quad -\infty < u < \infty.$$



The surface is determined by the parametric expressions:

$$y = Y(u,t), \qquad x = u - \hat{H}Y(u,t),$$

where $\hat{H}$ is the Hilbert transform: $\hat{H}f(u) = \pi^{-1}\int_{-\infty}^{+\infty} f(u')(u'-u)^{-1}du'.$

Equations of motion in conformal variables:

$$\Psi_t\left(1-\hat{H}Y_u\right)+\Psi_u\hat{H}Y_t+\hat{H}\left(Y_t\Psi_u-Y_u\Psi_t\right)=\hat{H}Y_u,$$

$$Y_t\left(1-\hat{H}Y_u\right)+Y_u\hat{H}Y_t=-\hat{H}\Psi_u,$$

where the function $\Psi(u,t)$ gives the value of the velocity potential on the surface.

The exact particular solutions:

$$\Psi=\pm\hat{H}Y=F(u\pm t),$$

where F is an arbitrary function.

Stability of nonlinear waves

Are the stationary wave solutions stable with respect to small perturbations?

Let us put $\Psi = F(u+t) + q(u,t), \quad Y = -\hat{H}F(u+t) + \delta(u,t),$

where q, δ are small-scale perturbations of the velocity potential and surface profile.

The linearized equations:

$$q_t - F'q_t + F'q_u - \hat{H}F' \cdot \hat{H}q_u + \hat{H}F' \cdot \hat{H}q_t = \hat{H}\delta_u - 2F'\hat{H}\delta_t + 2F'\hat{H}\delta_u,$$
$$\delta_t - F'\delta_t + F'\delta_u + \hat{H}F' \cdot \hat{H}\delta_u - \hat{H}F' \cdot \hat{H}\delta_t = -\hat{H}q_u.$$

We take $q \sim e^{iku - i\omega t}, \quad \delta \sim e^{iku - i\omega t}.$

The dispersion relation:

$$\omega = k, \qquad \omega = k - \frac{2k}{(1-F')^2 + (\hat{H}F')^2}.$$

Conclusion:

It has been shown that waves of arbitrary configuration in 3D geometry may propagate without distortion along the surface of a dielectric liquid in the direction of a horizontal electric field. This situation occurs for the high-permittivity liquids in the long-wave limit. A general solution of the equations of motion that describes the interaction of counter-propagating waves of a small but finite amplitude has been obtained. For 2D geometry, the stability of the nonlinear traveling waves is demonstrated using conformal variables.

Thank you
for attention!