Integrable 3D-systems of hydrodynamic type.

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References

1. Pavlov M.V., Odesskii A.V. and Sokolov V.V. *Classification of integrable Vlasov-type equations*, Teoret. and Mat. Fiz, 2008, **154**, no.**2**, 249–260.

2. Odesskii A.V. and Sokolov V. V. *Integrable pseudopotentials related to generalized hypergeometric functions*, arXiv:0803.0086, submitted to Selecta Math.

3. Odesskii A.V. and Sokolov V. V. *Integrable elliptic pseudopotentials*, arXiv:0810.3879, to be published Theor. and Math. Phys

4. Odesskii A.V. and Sokolov V. V. *Systems of Gibbons -Tsarev type and integrable 3-dimensional models*, arXiv:0906.3509

We consider dispersionless systems of PDEs for unknown functions $u_1(t,x,y), ..., u_n(t,x,y)$. Usually integrable systems of this kind admit dispersionless zero curvature representation. Namely, this system is equivalent to compatibility conditions for a pair of Hamilton-Jacobi equations

$$\Phi_y = A(\Phi_x, u_1, ..., u_n), \quad \Phi_t = B(\Phi_x, u_1, ..., u_n).$$

Sometimes such a pair is called *pseudopotential representation.*

Example. DKP. The system

$$u_y = v_x, \quad v_y = u_t - uu_x$$

possesses the pseudopotential representation

$$\Phi_y = \frac{\Phi_x^2}{2} + u, \quad \Phi_t = \frac{\Phi_x^3}{3} + u\Phi_x + v. \qquad \square$$

We denote Φ_x by p and $(u_1, ..., u_n)$ by $\mathbf{u}$.

Consider the simplest one-field case: $A = \psi(p, u)$.

The DKP- hierarchy provides the following two examples

$$\psi = \frac{p^2}{2} + u, \quad \text{and} \quad \psi = \log(p - u).$$

One explicit example more:

$$\psi = \sqrt{u\,(p^2 + c_1) + c_2}.$$

Pseudopotentials in the one-field case

Generic pseudopotentials $\psi(u,p)$ are given by

$$\psi_u = \frac{Q(\psi_p)}{\psi_{pp}}, \qquad \frac{\psi_{ppp}}{\psi_{pp}^2} = \frac{R(\psi_p)}{Q(\psi_p)}, \qquad (1)$$

where R and Q are polynomials in ψ_p such that $\deg R \leq 3$, $\deg Q \leq 4$. In the generic case (1) implies

$$\frac{\psi_{ppp}}{\psi_{pp}^2} = \frac{k_1}{\psi_p - b_1(u)} + ... + \frac{k_4}{\psi_p - b_4(u)}, \qquad (2)$$

$$b_i' = (1-k_i)\, a(u) \prod_{j\neq i} (b_i - b_j), \qquad i = 1, ..., 4, \qquad (3)$$

where k_i are any constants such that $k_1 + ... + k_4 = 3$, and $b_i = b_i(u)$. The function $a(u)$ can be chosen arbitrarily due to the admissible transformations $u \to s(u)$.

Let

$$S(h,\xi) = u(u-1)h_u + (1+s_1+s_2+s_3)(\xi-u)h,$$

where $h(u)$ is a solution of the standard hypergeometric equation

$$u(u-1)\,h(u)'' + [(\alpha+\beta+1)\,u-\gamma]\,h(u)' + \alpha\beta\,h(u) = 0,$$

where $s_1 = -\alpha$, $s_2 = \alpha-\gamma$, $s_3 = \gamma-\beta-1$. The parameters k_i are related to s_i by $k_i = s_i + 1$.

Define $P(h,\xi)$ by the formula

$$P(h,\xi) = \int_0^{\xi} S(h,z)(z-u)^{-s_1-1} z^{-s_2-1} (z-1)^{-s_3-1} dz.$$

Let h_1, h_2 be linearly independent solutions of the hypergeometric equation. The pseudopotential $A(p, u)$ can be defined in a parametric form by

$$A = P(h_1, \xi), \qquad p = P(h_2, \xi).$$

The next pseudopotential of hierarchy is given by $B = P_1(Z, \xi)$, where

$$Z(u, v) = \int_0^v (z - u)^{s_1} z^{s_2} (z - 1)^{s_3} dz.$$

An n-field generalization

Consider the following system of linear PDEs:

$$\frac{\partial^2 h}{\partial u_j \partial u_k} = \frac{s_j}{u_j - u_k} \cdot \frac{\partial h}{\partial u_k} + \frac{s_k}{u_k - u_j} \cdot \frac{\partial h}{\partial u_j}, \quad i, j = 1, ..., n, \qquad j \neq k,$$

and

$$\frac{\partial^2 h}{\partial u_j \partial u_j} = -\left(1 + \sum_{k=1}^{n+2} s_k\right) \frac{s_j}{u_j(u_j - 1)} \cdot h +$$

$$\frac{s_j}{u_j(u_j - 1)} \sum_{k \neq j}^{n} \frac{u_k(u_k - 1)}{u_k - u_j} \cdot \frac{\partial h}{\partial u_k} +$$

$$\left(\sum_{k \neq j}^{n} \frac{s_k}{u_j - u_k} + \frac{s_j + s_{n+1}}{u_j} + \frac{s_j + s_{n+2}}{u_j - 1}\right) \cdot \frac{\partial h}{\partial u_j}$$

for unknown function $h(u_1, \ldots, u_n)$. If $n = 1$, then this system coincides with the standard hypergeometric equation.

Proposition 1. This system is compatible for any constants $s_1, \ldots, s_{n+2}$. The dimension of the linear space $\mathcal{H}$ of solutions of the system equals $n + 1$.

We call any solution of the system *generalized hypergeometric function*.

For any $h \in \mathcal{H}$ we put

$$S(h) = \sum_{1 \leq i \leq n} u_i(u_i - 1)(\xi - u_1)...\hat{i}...(\xi - u_n)h_{u_i} +$$

$$(1 + \sum_{1 \leq i \leq n+2} s_i)(\xi - u_1)...(\xi - u_n)h.$$

Note that S is a polynomial of degree n in ξ. Similar (but more complicated) formulas define S in the case $\deg S < n$.

Define function $P(h, \xi)$ by

$$P(h, \xi) = \int_0^\xi S(h, z)(z - u_1)^{-s_1 - 1}...(z - u_n)^{-s_n - 1} \times$$

$$z^{-s_{n+1} - 1}(z - 1)^{-s_{n+2} - 1} dz.$$

Let $h_1, h_2 \in \mathcal{H}$ be linearly independent.

The pseudopotential $A(p, u_1, ..., u_n)$ can be defined in the parametric form by

$$A = P(h_1, \xi), \qquad p = P(h_2, \xi).$$

To define the second pseudopotential $B(p, u_1, ..., u_n)$ we need one more generalized hypergeometric function h_3: $\quad B = P(h_3, \xi)$.

Such pairs of pseudopotentials yield integrable 3D-systems of the form

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u}) \frac{\partial u_j}{\partial t} + \sum_{j=1}^{n} b_{ij}(\mathbf{u}) \frac{\partial u_j}{\partial y} + \sum_{j=1}^{n} c_{ij}(\mathbf{u}) \frac{\partial u_j}{\partial x} = 0,$$

where $i = 1, ..., l$. Here $l \geq n$. Integer $k = l - n$ is called the *defect* of the system. The defect is related to the degree of S.

Equations of the form

$$A_1 Z_{tt} + A_2 Z_{xt} + A_3 Z_{yt} + A_4 Z_{yy} + A_5 Z_{xy} + A_6 Z_{xx} = 0$$

where $A_i = A_i(Z_x, Z_y, Z_t)$, correspond to $n = 3, l = 4$.

Equations

$$F(Z_{tt}, Z_{xt}, Z_{yt}, Z_{yy}, Z_{xy}, Z_{xx}) = 0$$

correspond to $n = 5, l = 8$.

If

$$\Phi_y = A(p, \mathbf{u}), \quad \Phi_t = B(p, \mathbf{u}), \qquad \text{where} \quad p = \Phi_x$$

is a pseudopotential representation for some integrable 3D-system, then for any $p \in \mathbb{C}$ the point $\left(\frac{A_{ppp}}{A_{pp}^2}, A_p\right)$ belongs to an algebraic curve of genus g, whose coefficients depend on $\mathbf{u}$.

The pseudopotentials described above are related to rational curves.

We constructed also pseudopotentials and integrable systems related to the elliptic curve. For these systems $\mathbf{u} = (u_1, \ldots, u_n, \tau)$, where the parameter of the elliptic curve τ is also an unknown function in the system.

The pseudopotential $A_n(p, u_1, ..., u_n, \tau)$ is defined in the parametric form by

$$A_n = P_n(g_1, \xi), \qquad p = P_n(g_0, \xi),$$

where $g_1,\ g_0$ be linearly independent elliptic hypergeometric functions,

$$P_n(g, \xi) = \int_0^\xi S_n(g, z) e^{2\pi i r(\tau - z)} \times$$

$$\frac{\theta'(0)^{-s_1-\dots-s_n} \theta(u_1)^{s_1} \cdots \theta(u_n)^{s_n}}{\theta(z)^{-s_1-\dots-s_n} \theta(z-u_1)^{s_1} \cdots \theta(z-u_n)^{s_n}} dz,$$

and

$$S_n(g, \xi) = \sum_{1 \le \alpha \le n} \frac{\theta(u_\alpha)\theta(\xi - u_\alpha - \eta)}{\theta(u_\alpha + \eta)\theta(\xi - u_\alpha)} g_{u_\alpha} -$$

$$(s_1 + ... + s_n) \frac{\theta'(0)\theta(\xi - \eta)}{\theta(\eta)\theta(\xi)} g.$$

The pseudopotential is expressed in terms of solutions (elliptic hypergeometric functions of several variables) of the following linear system of PDEs:

$$g_{u_\alpha u_\beta} = s_\beta(\rho(u_\beta - u_\alpha) + \rho(u_\alpha + \eta) - \rho(u_\beta) - \rho(\eta))g_{u_\alpha} +$$

$$s_\alpha(\rho(u_\alpha - u_\beta) + \rho(u_\beta + \eta) - \rho(u_\alpha) - \rho(\eta))g_{u_\beta},$$

$$g_{u_\alpha u_\alpha} = s_\alpha \sum_{\beta\neq\alpha} (\rho(u_\alpha)+\rho(\eta)-\rho(u_\alpha-u_\beta)-\rho(u_\beta+\eta))g_{u_\beta} +$$

$$\Big(\sum_{\beta\neq\alpha} s_\beta \rho(u_\alpha - u_\beta) + (s_\alpha + 1)\rho(u_\alpha + \eta) +$$

$$s_\alpha \rho(-\eta) + (s_0 - s_\alpha - 1)\rho(u_\alpha) + 2\pi i r\Big)g_{u_\alpha} -$$

$$s_0 s_\alpha(\rho'(u_\alpha) - \rho'(\eta))g,$$

$$g_\tau = \frac{1}{2\pi i}\sum_\beta (\rho(u_\beta + \eta) - \rho(\eta))g_{u_\beta} - \frac{s_0}{2\pi i}\rho'(\eta)g$$

for a single function $g(u_1, \ldots, u_n, \tau)$.

Here $\eta = s_1 u_1 + ... + s_n u_n + r\tau + \eta_0, \quad s_0 = -s_1 - ... - s_n,$
where $s_1, ..., s_n, r, \eta_0$ are arbitrary constants, and

$$\theta(z) = \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha e^{2\pi i (\alpha z + \frac{\alpha(\alpha-1)}{2}\tau)}, \qquad \rho(z) = \frac{\theta'(z)}{\theta(z)}.$$

It turns out that the dimension of the space of solutions for the system equals $n+1$.

Some important examples of pseudopotentials A, B related to the Whitham averaging procedure for integrable dispersion PDEs, to the Frobenious manifolds, and to the WDVV-associativity equation were found by B. Dubrovin and I. Krichever.

In the case $s_1 = ... = s_n = r = 0$, $\eta_0 \to 0$ our pseudopotentials coincide with elliptic pseudopotentials constructed by Dubrovin and Krichever.

The explicit form of this system is given by

$$\sum_{i\neq j}((g_{1,u_j}g_{2,u_i}-g_{2,u_j}g_{1,u_i})\frac{u_j(u_j-1)u_{i,t_0}-u_i(u_i-1)u_{j,t_0}}{u_j-u_i}+$$

$$(1+s_1+...+s_{n+2})(g_1g_{2,u_j}-g_2g_{1,u_j})u_{j,t_0}+$$

$$\sum_{i\neq j}((g_{2,u_j}g_{0,u_i}-g_{0,u_j}g_{2,u_i})\frac{u_j(u_j-1)u_{i,t_1}-u_i(u_i-1)u_{j,t_1}}{u_j-u_i}+$$

$$(1+s_1+...+s_{n+2})(g_2g_{0,u_j}-g_0g_{2,u_j})u_{j,t_1}+$$

$$\sum_{i\neq j}((g_{0,u_j}g_{1,u_i}-g_{1,u_j}g_{0,u_i})\frac{u_j(u_j-1)u_{i,t_2}-u_i(u_i-1)u_{j,t_2}}{u_j-u_i}+$$

$$(1+s_1+...+s_{n+2})(g_0g_{1,u_j}-g_1g_{0,u_j})u_{j,t_2}=0.$$

Pseudopotentials of defect $k > 0$

To define pseudopotentials of defect k, we fix k linearly independent generalized hypergeometric functions $h_1, ..., h_k \in \mathcal{H}$. For any $g \in \mathcal{H}$ define $S_k(g, \xi)$ by

$$S_k(g,\xi) = \frac{1}{\Delta} \sum_{1 \le i \le n-k+1} u_i(u_i - 1)(\xi - u_1) \times ...\hat{i}... \times(\xi - u_{n-k+1})\Delta_i(g).$$

Here

$$\Delta = \det \begin{pmatrix} h_1 & ... & h_k \\ h_{1,u_{n-k+2}} & ... & h_{k,u_{n-k+2}} \\ .... & ... & .... \\ h_{1,u_n} & ... & h_{k,u_n} \end{pmatrix},$$

$$\Delta_i(g) = \det \begin{pmatrix} g & h_1 & \dots & h_k \\ g_{u_i} & h_{1,u_i} & \dots & h_{k,u_i} \\ g_{u_{n-k+2}} & h_{1,u_{n-k+2}} & \dots & h_{k,u_{n-k+2}} \\ \dots\dots\dots & \dots & \dots & \dots\dots\dots \\ g_{u_n} & h_{1,u_n} & \dots & h_{k,u_n} \end{pmatrix}.$$

It is clear that $S_{n,k}(g,\xi)$ is a polynomial in ξ of degree $n-k$.

Example 3. In the simplest case $n = 2$, $k = 1$ we have

$$S_1(g,\xi) = u_1(u_1 - 1)(\xi - u_2)\frac{gh_{1,u_1} - g_{u_1}h_1}{h_1} +$$

$$u_2(u_2 - 1)(\xi - u_1)\frac{gh_{1,u_2} - g_{u_2}h_1}{h_1}.$$

Then the function $P_k(g, \xi)$ is defined by

$$P_k(g,\xi) = \int_0^{\xi} S_k(g,z)(z-u_1)^{-s_1-1}...(z-u_{n-k+1})^{-s_{n-k+1}-1}$$

$$\times(z-u_{n-k+2})^{-s_{n-k+2}}...(z-u_n)^{-s_n} z^{-s_{n+1}-1}(z-1)^{-s_{n+2}-1} dz.$$