Kinetic equation for a soliton gas

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Motivation

Description of "integrable turbulence".

- V.E. Zakharov, Turbulence in integrable systems, *Stud. Appl. Math.* (2009)

A particular important aspect: theory of soliton gases.

- V.E. Zakharov, Kinetic equation for solitons, *JETP* (1971)

Here we consider only **strongly integrable** systems (like KdV, NLS etc.)

From N-solitons/N-gap potentials to a soliton gas

N-solitons: two approaches

- IST: reflectionless potentials (N-soliton solutions);
- Finite-gap theory: closing all spectral bands in the N-gap potential leads to the N-soliton.

Soliton gas: $N \to \infty$; a generalised reflectionless potential (Marchenko) with shift invariant probability measure on it.

- IST:
  - **Gurevich, Zybin et. al.** (2000, 2002): stochastic version of the Lax-Levermore approach (KdV and defocusing NLS)
  - **Kotani** (2008): KdV flow on generalized reflectionless potentials (Sato 's Grassmanian approach)
- Finite-gap theory: El, Krylov, Molchanov and Venakides (1999): soliton gas as the thermodynamic limit of finite-gap potentials:

$$N \to \infty\,, \quad L \to \infty \quad N/L \sim \sum_{j=1}^{N} k_j = O(1)$$

From the Whitham equations to kinetic equation for solitons

- Assuming the existence of the invariant spectral measure characterising spatially **homogeneous** soliton gas, we want to describe **slow evolution** of this measure for an **inhomogeneous** gas, i.e. to derive **kinetic equation** for solitons.

- Similarity with the formulation of the **Whitham modulation theory** (but with the **averaging over ensemble** rather than period!). *P.D. Lax (1991): The zero dispersion limit, a deterministic analogue of turbulence.*

- Hence our interest in the **thermodynamic limit** of the Whitham equations.

Outline

- **Kinetic equation for solitons as the thermodynamic limit of the Whitham equations**
- **Generalised kinetic equation**
- **Hydrodynamic reductions and integrability**
- **Conclusions**

Also, if time permits;

- Moments of the wave field
- Further research directions

The Whitham equations (KdV case)

The generating equation for the KdV-Whitham system is *(Flaschka, Forest, McLaughlin 1979)*

$$(dp_N)_t = (dq_N)_x \, ,$$

where dp_N and dq_N are the quasimomentum and quasienergy differentials on the hyperelliptic Riemann surface of genus N:

$$R^2(\lambda) = \prod_{j=1}^{2N+1} (\lambda - \lambda_j) \, , \qquad \lambda \in \mathbb{C}, \quad \lambda_j \in \mathbb{R} \, .$$

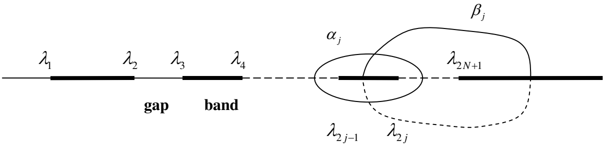
Asymptotic behaviour near $\lambda = -\infty$:

$$-\lambda \gg 1 \;\; : \qquad dp_N \sim -\frac{d\lambda}{(-\lambda)^{1/2}} \, , \qquad dq_N \sim (-\lambda)^{1/2} d\lambda$$

The Whitham equations

The differentials dp_N and dq_N are uniquely defined by the normalisation:

$$\oint_{\beta_i} dp_N(\lambda) = 0\,, \qquad \oint_{\beta_i} dq_N(\lambda) = 0\,, \qquad i = 1, \ldots, N\,.$$



The fundamental wavenumbers k_j and frequencies ω_j are found as

$$k_j(\lambda_1, \ldots, \lambda_{2N+1}) = \oint_{\alpha_j} dp_N(\lambda)\,, \quad \omega_j(\lambda_1, \ldots, \lambda_{2N+1}) = \oint_{\alpha_j} dq_N(\lambda)\,,$$

Whitham equations as the equations for the spectral measure

Assume that the finite-band part of the spectrum λ lies in $[-1, 0]$.
We integrate the Whitham system $\partial_t dp_N(\lambda) = \partial_x dq_N(\lambda)$ on the real line from -1 to $-\eta^2 \in [-1, 0]$ and take the real part:

$$\partial_t \mathcal{N}_N(-\eta^2) = \partial_x \mathcal{V}_N(-\eta^2)\,.$$

Here

$$\mathcal{N}_N(\lambda) = \frac{1}{\pi} Re \int_{-1}^{\lambda} dp_N(\lambda')$$

is the **integrated density of states** (Johnson & Moser 1982), and
$\mathcal{V}_N(\lambda) = \frac{1}{\pi} Re \int_{-1}^{\lambda} dq_N(\lambda')$ – its temporal analog.

Importantly, $d\mathcal{N}_N(-\eta^2)$ is a **measure**.

Thermodynamic limit

The total density of states

$$\rho_N = \frac{1}{\pi} Re \int_{-1}^{0} dp_N(\lambda') = \frac{1}{2\pi} \sum_{j=1}^{N} k_j$$

In the thermodynamic limit $\lim_{N\to\infty} \rho_N = O(1)$.
This is achieved by the following (thermodynamic) spectral scaling

$$|\text{gap}_j| \sim \frac{1}{\varphi(\eta_j)N} \qquad |\text{band}_j| \sim \exp\{-\gamma(\eta_j)N\}, \quad j = 1, \dots, N$$

where $\phi(\eta)$, $\gamma(\eta)$ are some continuous positive functions on $[0,1]$.

- Venakides (1989) The continuum limit of theta functions;
- El, Krylov, Molchanov & Venakides (1999) Soliton turbulence as the thermodynamic limit od soliton lattices

Note: $|\text{band}_j|/|\text{gap}_j| \to 0$ as $N \to \infty$ $\forall j$ i.e. **“infinite-soliton” limit**.

The thermodynamic limit of the Whitham equations

- The modulation system $\partial_t \mathcal{N}_N(-\eta^2) = \partial_x \mathcal{V}_N(-\eta^2)$
- In the thermodynamic limit as $N \to \infty$:
  - $d\mathcal{N}_N \to \pi f(\eta) d\eta > 0$, $\quad d\mathcal{V}_N \to -\pi f(\eta) s(\eta) d\eta$
  - $s(\eta)$ and $f(\eta)$ are related via:

$$s(\eta) = -4\eta^2 + \frac{1}{\eta} \int_0^1 \ln \left| \frac{\eta - \mu}{\eta + \mu} \right| f(\mu)[s(\eta) - s(\mu)] d\mu, \qquad (1)$$

Now, we postulate that on a larger scale, $\Delta x, \Delta t \gg 1$:

$$f(\eta) = f(\eta, x, t), \qquad s(\eta) = s(\eta, x, t)$$

Then the modulation system transforms into

$$f_t = (fs)_x \,, \qquad (2)$$

Equations (2), (1) form a closed system : the kinetic equation for the KdV soliton gas of **finite density** (*El 2003*)

Kinetic equation for solitons: small-density expansion

We replace $s \to -s$. Now s is the velocity of soliton gas.

$$f_T + (fs)_X = 0\,, \tag{1}$$

$$s(\eta) = 4\eta^2 + \frac{1}{\eta}\int_0^\infty \ln\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)[s(\eta)-s(\mu)]d\mu, \tag{2}$$

The small-density, $\rho = \int_0^\infty f d\eta \ll 1$, expansion of (2), yields

$$s(\eta) = 4\eta^2 + \frac{1}{\eta}\int_0^\infty \ln\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)[4\eta^2-4\mu^2]d\mu + \mathcal{O}(\rho^2), \tag{3}$$

– the velocity of a 'trial' soliton in a rarefied soliton gas (Zakharov 1971). So Eqs. (1), (2) represent a generalisation of Zakharov's kinetic equation for a rarefied soliton gas to the case of the gas of finite density.

Generalised kinetic equations for soliton gases with elastic collisions *El & Kamchatnov (PRL 2005)*

- Main ingredients: (i) **the speed of a free soliton** $S(\eta)$ and (ii) **the phase shift** $\Delta x_{\eta,\mu} = G(\eta,\mu)$ due to the soliton-soliton collision.
- Introduce the spectral distribution function $f(\eta) \equiv f(\eta, x, t)$ and the mean speed of a 'trial' η- soliton $s(\eta) \equiv s(\eta, x, t)$
- Then the self-consistent definition of the soliton velocity $s(\eta)$ in a dense soliton gas with the spectral distribution $f(\eta)$ is given by the integral equation

$$s(\eta) = S(\eta) + \int_0^\infty G(\eta,\mu)[s(\eta) - s(\mu)]f(\mu)d\mu \,,$$

- Isospectrality implies the conservation equation for the spectral distribution function $f(\eta, x, t)$:

$$f_t + (sf)_x = 0 \,.$$

Hydrodynamic reductions of the kinetic equation.

$$f_t+(sf)_x = 0\,, \quad s(\eta,x,t) = S(\eta)+\int_0^\infty G(\eta,\mu)[s(\eta,x,t)-s(\mu,x,t)]f(\mu,x,t)d\mu \tag{1}$$

We introduce $u(\eta,x,t)=\eta f(\eta,x,t)$, $v(\eta,x,t)=-s(\eta,x,t)$ and consider N-component 'cold-gas' *ansatz*

$$u=\sum_{i=1}^{N} u^i(x,t)\delta(\eta-\eta_i)\,,$$

which reduces (1) to a system of N hydrodynamic conservation laws,

$$\partial_t u^i = \partial_x(u^i v^i), \qquad i=1,\dots,N\,,$$

where the velocities $v^i=v(\eta_i,x,t)$ and the 'densities' u^i are related via

$$v^i=\xi_i+\sum_{k=1}^{N}\epsilon_{ik}u^k(v^k-v^i)\,, \qquad \epsilon_{ik}=\epsilon_{ki},$$

$$\xi_i=S(\eta_i)\,, \qquad \epsilon_{ik}=\frac{1}{\eta_i\eta_k}G(\eta_i,\eta_k)>0\,, \qquad v^i=s(\eta_i).$$

Hydrodynamic reductions: $N = 2$

For $N = 2$ the system of hydrodynamic laws assumes the form

$$\partial_t u^1 = \partial_x(u^1 v^1), \qquad \partial_t u^2 = \partial_x(u^2 v^2)$$

$$u^1 = \frac{1}{\epsilon_{12}} \frac{v^2 - \xi_2}{v^1 - v^2}, \qquad u^2 = \frac{1}{\epsilon_{12}} \frac{v^1 - \xi_1}{v^2 - v^1}.$$

Passing to the *Riemann invariants* we obtain

$$v_t^1 = v^2 v_x^1, \qquad v_t^2 = v^1 v_x^2. \tag{1}$$

The system (1) is linearly degenerate, i.e. its characteristic velocities do not depend on the corresponding Riemann invariants.

What about $N > 2$?

Hydrodynamic reductions: $N \geq 3$.

Theorem (El, Kamchatnov, Pavlov & Zykov 2008)

N-component hydrodynamic type system

$$\partial_t u^i = \partial_x(u^i v^i), \qquad i = 1, \dots, N\,,$$

$$v^i = \xi_i + \sum_{k=1}^{N} \epsilon_{ik} u^k (v^k - v^i), \qquad \epsilon_{ik} = \epsilon_{ki},$$

where $\xi_1, \xi_2, \dots, \xi_N$ are constants and ϵ is a constant symmetric matrix, $\epsilon_{ik} = \epsilon_{ki}$, is:

- *diagonalizable*
- *linearly degenerate,*
- *semi-Hamiltonian (i.e. integrable - Tsarev 1985, 1991),*

for any N.

The proof is based on the theory of integrable linearly degenerate hydrodynamic type systems developed by Pavlov (1987) and Ferapontov (1991).

Linearly degenerate hydrodynamic type systems

Definition. A diagonal hydrodynamic type system $r_t^i = V^i(\mathbf{r})r_x^i$ is called *linearly degenerate* if $\partial_i V^i = 0 \ \forall i$. ($\partial_k \equiv \partial/\partial r^k$)

A *semi-Hamiltonian* (i.e. integrable) linearly degenerate hydrodynamic type system $r_t^i = V^i(\mathbf{r})r_x^i$ is characterised by the so-called Stäkel matrix

$$\Delta = \begin{pmatrix} \phi_1^1(r^1) & \dots & \phi_N^1(r^N) \\ \dots & \dots & \dots \\ \phi_1^{N-2}(r^1) & & \phi_N^{N-2}(r^N) \\ \phi_1^{N-1}(r^1) & & \phi_N^{N-1}(r^N) \\ 1 & \dots & 1 \end{pmatrix}$$

where $\phi_k^i(r^k)$ are certain functions, so that (Ferapontov 1991)

$$V^i(\mathbf{r}) = \frac{\det \Delta_i^{(2)}}{\det \Delta_i^{(1)}},$$

where $\Delta_i^{(k)}$ is the matrix Δ without k-th row and i-th column.

Linearly degenerate conservation laws

It follows from Pavlov (1987) and Ferapontov (1991) that the system of conservation laws

$$u^i_t = (u^i v^i)_x, \quad v^i = v^i(\mathbf{u}(\mathbf{r})) \quad i = 1, \dots, N$$

is a semi-Hamiltonian linearly degenerate hydrodynamic type system iff the densities u^i and velocities $v^i(\mathbf{u})$ admit the representations

$$u^i = \frac{\det \Delta^{(1)}_i}{\det \Delta}(-1)^{i+1} P_i(r^i), \qquad v^i = \frac{\det \Delta^{(2)}_i}{\det \Delta^{(1)}_i}$$

in terms of the Stäkel matrix Δ via N functions r^k; here $P_i(r^i)$ are arbitrary functions.

For the N - component hydrodynamic reductions of the kinetic equation it was proved in (El, Kamchatnov, Pavlov & Zykov 2008) that such a parametrization exists for any N.

Hence: integrability of the 'cold-gas' hydrodynamic reductions for any N.

N=3: explicit formulae (*El, Kamchatnov, Pavlov & Zykov, 2008*).

The three-component 'cold-gas' hydrodynamic reduction of the nonlocal kinetic equation

$$\partial_t u^i = \partial_x (u^i v^i), \qquad i = 1, 2, 3\,,$$

$$v^i = \xi_i + \sum_{k=1}^{3} \epsilon_{ik} u^k (v^k - v^i) \qquad \epsilon_{ik} = \epsilon_{ki}.$$

has the Riemann invariant representation

$$\partial_t r^j = V^j(\mathbf{r}) \partial_x r^j\,, \qquad j = 1, 2, 3,$$

where

$$V^1 = \frac{\zeta_2 r^2 - \zeta_3 r^3}{r^2 - r^3}, \quad V^2 = \frac{\zeta_3 r^3 - \zeta_1 r^1}{r^3 - r^1}, \quad V^3 = \frac{\zeta_1 r^1 - \zeta_2 r^2}{r^1 - r^2}$$

$$\zeta_1 = \frac{\xi_3 \epsilon_{12} - \xi_2 \epsilon_{13}}{\epsilon_{12} - \epsilon_{13}}, \quad \zeta_2 = \frac{\xi_1 \epsilon_{23} - \xi_3 \epsilon_{12}}{\epsilon_{23} - \epsilon_{12}}, \quad \zeta_3 = \frac{\xi_1 \epsilon_{23} - \xi_2 \epsilon_{13}}{\epsilon_{23} - \epsilon_{13}}$$

N=3: explicit formulae (*El, Kamchatnov, Pavlov & Zykov, 2008*).

The Riemann invariants r^1, r^2, r^3 are expressed in terms of the densities u^1, u^2, u^3 as

$$
\begin{aligned}
r^1 &= \frac{(\epsilon_{12}-\epsilon_{13})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{23})}{[(\xi_3-\xi_1)\epsilon_{12}+(\xi_1-\xi_2)\epsilon_{13}]u^1-(\xi_2-\xi_3)(\epsilon_{12}u^2+\epsilon_{13}u^3+1)},\\
r^2 &= \frac{(\epsilon_{23}-\epsilon_{12})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{13})}{[(\xi_1-\xi_2)\epsilon_{23}+(\xi_2-\xi_3)\epsilon_{12}]u^2-(\xi_3-\xi_1)(\epsilon_{12}u^1+\epsilon_{23}u^3+1)},\\
r^3 &= \frac{(\epsilon_{13}-\epsilon_{23})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{12})}{[(\xi_2-\xi_3)\epsilon_{13}+(\xi_3-\xi_1)\epsilon_{23}]u^3-(\xi_1-\xi_2)(\epsilon_{13}u^1+\epsilon_{23}u^2+1)}.
\end{aligned}
$$

Conclusions

- The thermodynamic limit of the Whitham equations associated with hyperelliptic Riemann surfaces leads to the kinetic equations for the corresponding soliton gases.
- The original Zakharov (1971) prescription for the determination of the velocity of a trial soliton in a rarefied soliton can be directly extended to the case of the soliton gas of finite density.
- N-component cold gas hydrodynamic reductions of the generalized kinetic equation for solitons represent linearly degenerate semi-Hamiltonian (integrable) systems of hydrodynamic type for any N.

References

[1] El, G.A., Krylov A.L., Molchanov, S.A. and Venakides, S., "Soliton turbulence as a thermodynamic limit of stochastic soliton lattices", Advances in Nonlinear Mathematics and Science, *PHYSICA D*, 152/153, 2001, pp 653-664.

[2] El, G.A., "The thermodynamic limit of the Whitham equations", *PHYS LETT A*, 311, 2003, pp 374-383.

[3] El, G.A. and Kamchatnov, A.M., "Kinetic equation for a dense soliton gas", *PHYS REV LETT* 95, 2005, Art No 204101, 4 pp.

[4] El, G.A., Kamchatnov, A.M., Pavlov, M.V., and Zykov, S.A., "Nonlocal kinetic equation: integrable hydrodynamic reductions, symmetries and exact solutions" nlin. arXive 2008.

Wave field moments in a soliton gas: the KdV case

We have obtained that in the thermodynamic limit $d\mathcal{N}_N \to \pi f(\eta, x, t) d\eta$. On the other hand, it is well known that the integrated density of states $\mathcal{N}_N(\lambda)$ is the generating function for the averaged Kruskal integrals $I_k^{(N)}$:

$$\mathcal{N}_N(\lambda) = 2\sqrt{-\lambda} + \sum_{k=0}^{\infty} \frac{I_k^{(N)}}{(-2\lambda)^k}, \qquad -\lambda \gg 1,$$

$$I_0^{(N)} = \lim_{L\to\infty} \frac{1}{L} \int_0^L u_N(x) dx, \quad I_1^{(N)} = \lim_{L\to\infty} \frac{1}{L} \int_0^L u_N^2(x) dx, \dots$$

Then the thermodynamic limit of Kruskal integrals is found as

$$I_k^{(N)} \to \frac{2^{2k+2}}{2k+1} (-1)^{k+1} \int_0^{\infty} \eta^{2k+1} f(\eta) d\eta, \qquad k = 0, 1, 2, \dots$$

Wave field moments in a soliton gas: the KdV case

For the two first moments we have

$$\overline{u}(x,t) = 4\int_0^\infty \eta f(\eta,x,t)d\eta\,, \qquad \overline{u^2}(x,t) = \frac{16}{3}\int_0^\infty \eta^3 f(\eta,x,t)d\eta$$

Important restriction:

$$\sigma = \overline{u^2} - \overline{u}^2 \geq 0 \tag{1}$$

For instance, for a one-component soliton gas $f = f_0(x,t)\delta(\eta-\eta_0)$, so

$$\overline{u} = 4\eta_0 f_0(x,t)\,, \qquad \overline{u^2} = \frac{16}{3}\eta_0^3 f_0(x,t)\,.$$

i.e. the variance

$$\sigma = 16 f_0 \eta_0^2 (\frac{\eta_0}{3} - f_0)\,.$$

Now one can see that (1) imposes a restriction on the possible density values f_0 for a one-component soliton gas with a given η_0:

$$f_0 < f_{cr} = \frac{\eta_0}{3}.$$

Critical density for a soliton gas.

Future research

- **Connection with hydrodynamic chains and 2+1 dispersionless systems.**
  Motivation: The collisionless Boltzmann kinetic equation for the distribution function $f(p, x, t)$

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial x} - \left(\int \frac{\partial f}{\partial x} dp\right)\frac{\partial f}{\partial p} = 0$$

  is equivalent to the Benney moment chain (*Zakharov 1981, Gibbons 1981*)

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + nA_{n-1}\frac{\partial A_0}{\partial x} = 0\,, \quad A_n = \int p^n f dp$$

  and to the dispersionless KP-II equation (*Kupershmidt & Manin ??, Gibbons 1984, Kodama 1988*)

$$(u_t + 6uu_x)_x + u_{yy} = 0\,.$$

- **Construction of physically relevant exact solutions.**
- **PDF, correlation function.**

Appendix 1. N=3: Exact solutions.

1. Similarity solutions

The family of the similarity solutions.

$$r^i = \frac{1}{t^\alpha} l^i\left(\frac{x}{t}\right), \quad i = 1, 2, 3,$$

is implicitly specified by the algebraic system

$$\begin{aligned}
\frac{x}{t} &= c_1\zeta_1(l^1)^\gamma + c_2\zeta_2(l^2)^\gamma + c_3\zeta_3(l^3)^\gamma\,, \\
-1 &= c_1(l^1)^\gamma + c_2(l^2)^\gamma + c_3(l^3)^\gamma\,, \\
0 &= c_1(l^1)^{\gamma-1} + c_2(l^2)^{\gamma-1} + c_3(l^3)^{\gamma-1}\,,
\end{aligned}$$

where $\gamma = -1/\alpha$ and c_1, c_2, c_3 are arbitrary constants.

2. Quasi-periodic solutions

The family of the quasi-periodic (3 periods) solution is implicitly specified by the system

$$
\begin{aligned}
x &= \zeta_1 \int^{r^1} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \zeta_2 \int^{r^2} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \zeta_3 \int^{r^3} \frac{\xi d\xi}{\sqrt{R_7(\xi)}}, \\
-t &= \int^{r^1} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \int^{r^2} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \int^{r^3} \frac{\xi d\xi}{\sqrt{R_7(\xi)}}, \\
0 &= \int^{r^1} \frac{d\xi}{\sqrt{R_7(\xi)}} + \int^{r^2} \frac{d\xi}{\sqrt{R_7(\xi)}} + \int^{r^3} \frac{d\xi}{\sqrt{R_7(\xi)}},
\end{aligned}
$$

where

$$R_7(\xi) = \prod_{m=1}^{7} (\xi - E_m),$$

$E_1 < E_2 < \cdots < E_7$ are arbitrary real constants.

Appendix 2. Kinetic equation for the focusing NLS soliton gas

The focusing nonlinear Schrödinger (NLS) equation has the form

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

Then the kinetic equation for a dense gas of bright NLS solitons is

$$f_T + (sf)_X = 0\,,$$

$$s(\alpha,\gamma) = -4\alpha + \frac{1}{2\gamma}\int_{-\infty}^{\infty}\int_0^{\infty} \ln\left|\frac{\lambda-\overline{\mu}}{\lambda-\mu}\right|^2 f(\xi,\eta)[s(\alpha,\gamma)-s(\xi,\eta)]d\xi d\eta\,.$$

Here

$$f \equiv f(\alpha,\gamma;X,T), \quad s \equiv s(\alpha,\gamma;X,T)$$

$$\lambda = \alpha + i\gamma\,, \qquad \mu = \xi + i\eta$$

Appendix 2. Kinetic equation for the focusing NLS soliton gas

N=2: collision of two cold soliton gases.
Let $\rho_1(x,t) = \rho_{10} = constant_1, \quad \rho_2(x,t) = \rho_{10} = constant_2$.

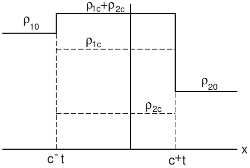
No continuous solution.

Two **hyperbolic** conservation laws available:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial (s_1 \rho_1)}{\partial x} = 0, \quad \frac{\partial \rho_2}{\partial t} + \frac{\partial (s_2 \rho_2)}{\partial x} = 0,$$

$$s_1 = s_1(\rho_1, \rho_2) \qquad s_2 = s_2(\rho_1, \rho_2)$$

Weak solution: three constant states separated by two strong discontibuities.



A2. Kinetic equation for the focusing NLS soliton gas

N=2: collision of two cold soliton gases.

Jump conditions:

$$-c^-[\rho_{1c} - \rho_1^-] + [\rho_{1c}s_{1c} - \rho_1^- s_1^-] = 0,$$
$$-c^-[\rho_{2c} - \rho_2^-] + [\rho_{2c}s_{2c} - \rho_2^- s_2^-] = 0,$$

$$-c^+[\rho_{1c} - \rho_1^+] + [\rho_{1c}s_{1c} - \rho_1^+ s_1^+] = 0,$$
$$-c^+[\rho_{2c} - \rho_2^+] + [\rho_{2c}s_{2c} - \rho_2^+ s_2^+] = 0.$$

As a result, the densities and velocities of the components in the interaction region are

$$\rho_{1c} = \frac{\rho_{10}(1-\kappa\rho_{20})}{1-\kappa^2\rho_{10}\rho_{20}}, \qquad \rho_{2c} = \frac{\rho_{20}(1-\kappa\rho_{10})}{1-\kappa^2\rho_{10}\rho_{20}}.$$

$$s_{1c} = 4\alpha\frac{1-\kappa(\rho_{1c}-\rho_{2c})}{1-\kappa(\rho_{1c}+\rho_{2c})}, \quad s_{2c} = -4\alpha\frac{1+\kappa(\rho_{1c}-\rho_{2c})}{1-\kappa(\rho_{1c}+\rho_{2c})}.$$

The speeds of the 'shocks' :

$$c^- = -4\alpha\frac{1+\kappa\rho_{10}}{1-\kappa\rho_{10}}, \qquad c^+ = 4\alpha\frac{1+\kappa\rho_{20}}{1-\kappa\rho_{20}}.$$