The Fifth International Conference
"Solitons, Collapses and Turbolence"

dedicated to Zakharov's 70 anniversary

Chernogolovka, Russia. August 2 - 7, 2009

Integrable PDEs arising as commutation of vector fields: Cauchy problem, longtime behaviour, particular solutions and multidimensional wave breaking

S. V. Manakov and P. M. Santini

We have recently solved the inverse scattering problem for one-parameter families of vector fields. This theory has been used to study integrable and applicative PDEs arising as commutation of vector field, like the heavenly, the dispersionless Kadomtsev-Petviashvili (dKP) equation and the 2D dispersionless Toda (2ddT) equation. In particular, the following issues have been successfully investigated: i) the Cauchy problem; ii) the longtime behavior of the solutions; iii) distinguished classes of implicit solutions; iv) whether localized data break at finite time and, if they do it, the geometric and analytic details of such a two-dimensional wave breaking.

The commutation of linear, first order, partial differential operators with scalar coefficients (vector fields) $[\hat{L}_1, \hat{L}_2] = 0$, or $\hat{L}_1\psi = \hat{L}_2\psi = 0$ leads to integrable quasi-linear PDEs in arbitrary dimensions (Zakharov-Shabat '79).

Basic examples:

1) A vector PDE in $4 + N$ dimensions (N arbitrary) (MS'06):

$$\vec{U}_{t_1 z_2} - \vec{U}_{t_2 z_1} + \left(\vec{U}_{z_1} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_2} - \left(\vec{U}_{z_2} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_1} = \vec{0}, \qquad (1)$$

$$\begin{array}{l} \hat{L}_i := \partial_{t^i} + \lambda \partial_{z^i} + \vec{U}_{z_i} \cdot \nabla_{\vec{x}}, \qquad i = 1, 2 \\ \nabla_{\vec{x}} = (\partial_{x^1}, .., \partial_{x^N}), \quad \vec{U} = (U^1, .., U^N) \end{array} \qquad (2)$$

2) Its basic reduction, the (4-dimensional) second heavenly equation of Plebanski:

$$\theta_{zy} - \theta_{tx} + \theta_{xy}^2 - \theta_{xx}\theta_{yy} = 0, \quad \theta = \theta(x, y, z, t) \qquad (3)$$

Its Hamiltonian formulation:

$$\begin{array}{ll} \hat{L}_1 = \partial_z - \{H_1, \cdot\}_{\vec{x}}, & \hat{L}_2 = \partial_t - \{H_1, \cdot\}_{\vec{x}}, \\ (H_1, H_2) := \nabla_{\vec{x}}\theta, & \vec{x} = (x, y). \end{array} \qquad (4)$$

Applications: self-dual Einstein fields.

3) The dKP system (MS'06):

$$u_{xt} + u_{yy} = -(uu_x)_x - v_x u_{xy} + v_y u_{xx}, \;\; u, v \in \mathbb{R}, \quad x, y, t \in \mathbb{R},$$
$$v_{xt} + v_{yy} = -uv_{xx} - v_x v_{xy} + v_y v_{xx}$$

$$v = 0: \;\; u_{xt} + u_{yy} + (uu_x)_x = 0, \qquad u = u(x, y, t) \quad \text{dKP}$$
$$u = 0: \;\; v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, \;\; v = v(x, y, t) \text{Pavlov} \tag{5}$$

$$\begin{aligned} \hat{L}_1 &\equiv \partial_y + (\lambda + v_x)\partial_x - u_x \partial_\lambda, \\ \hat{L}_2 &\equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y)\partial_x + (-\lambda u_x + u_y)\partial_\lambda. \end{aligned} \tag{6}$$

Hamiltonian formulation of the dKP equation (Zakharov 94, Krichever 94):

$$\begin{aligned} \hat{L}_1 &= \partial_y + \{H_1, \cdot\}_{(\lambda,x)}, \\ \hat{L}_2 &= \partial_t + +\{H_2, \cdot\}_{(\lambda,x)}, \end{aligned} \tag{7}$$

$$H_{1t} - H_{2y} + \{H_2, H_1\}_{(\lambda,x)} = 0,$$

$$H_1 = \tfrac{\lambda^2}{2} + u(x, y), \qquad H_2 = \tfrac{\lambda^3}{3} + \lambda u - \partial_x^{-1} u_y, \tag{8}$$
$$\{f, g\}_{(\lambda,x)} \equiv f_\lambda g_x - f_x g_\lambda,$$

Elegant integration scheme for PDEs associated with Hamiltonian vector fields (Krichever 94).
Applications: small amplitude, nearly one-dimensional waves in shallow water, with negligeable dispersion in the longitudinal direction, nonlinear acoustics of confined beams, unsteady motion in transonic flow, ..

4) The 2ddToda equation (Finley-Plebanski), (Boyer-Finley):

$$\phi_{\zeta_1\zeta_2} = \left(e^{\phi_{\zeta_3}}\right)_{\zeta_3}, \quad \phi = \phi(\zeta_1, \zeta_2, \zeta_3). \tag{9}$$

i) continuous limit of 2D-Toda lattice: $n\epsilon \to \zeta_3$ (space-like variable). ii) If $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}$: continuous limit of the Laplace transformation of a conjugate net (Darboux), (time (ζ_3)-evolution of a conjugate net);
iii) $\zeta_1 = (x+iy)/2,\ \zeta_2 = (x-iy)/2,\ \zeta_3 = t$: a nonlinear wave equation relevant in the theory of "heavens", in Einstein-Weyl geometries; string equ. solutions of it are relevant in Hele-Shaw flows:

$$\phi_{xx} + \phi_{yy} = \left(e^{\phi_t}\right)_t \quad \text{or} \quad \varphi_{xx} + \varphi_{yy} = (e^{\varphi})_{tt},\ \varphi = \phi_t. \tag{10}$$

Commuting Hamiltonian vector fileds: $[\hat{L}_1, \hat{L}_2] = 0$ (Takasaki-Takebe 91):

$$\begin{aligned} \hat{L}_1 &= \partial_{\zeta_1} + \lambda v \partial_t + \left(-\lambda v_t + \tfrac{\phi_{\zeta_1 t}}{2}\right)\lambda\partial_\lambda, \\ \hat{L}_2 &= \partial_{\zeta_2} + \lambda^{-1} v \partial_t + \left(\lambda^{-1} v_t - \tfrac{\phi_{\zeta_2 t}}{2}\right)\lambda\partial_\lambda, \\ v &= e^{\frac{\phi_t}{2}} \end{aligned} \tag{11}$$

Hamiltonian IST formulation:

$$\begin{aligned} &\psi_{\zeta 1} + \{\mathcal{H}_1, \psi\}_{(\lambda,t)} = 0, \quad \psi_{\bar{\zeta}2} + \{\mathcal{H}_2, \psi\}_{(\lambda,t)} = 0, \\ &\mathcal{H}_{1\zeta_2} - \mathcal{H}_{2\zeta_1} - \{\mathcal{H}_1, \mathcal{H}_2\}_{(\lambda,t)} = 0, \\ &\mathcal{H}_1 = \lambda e^{\phi_t/2} - \tfrac{\phi_z}{2}, \quad \mathcal{H}_2 = -\lambda^{-1} v + \tfrac{\phi_{\bar{z}}}{2}, \\ &\{f, g\}_{(\lambda,t)} := \lambda(f_\lambda g_t - f_t g_\lambda). \end{aligned} \tag{12}$$

Nearly one-dimensional and weakly nonlinear waves of 2ddT are described by the universal dKP model:

$$\phi(x,y,t) = \epsilon\ u(x-t, \epsilon^{1/2}y, \epsilon t),$$
$$u_{\xi\tau} + u_{\eta\eta} + (u_\xi^2)_\xi = 0, \quad \xi = x - t, \quad \eta = \epsilon^{1/2}y, \quad \tau = \epsilon t. \tag{13}$$

we expect that, at least asymptotically, the 2ddT dynamics will be similar to that of dKP.

Effectiveness and simplicity of the approach:
Nonlinear RH dressing

Nonlinearity: The space of eigenfunctions is a ring (an arbitrary differentiable function of the basic eigenfunctions is an eigenfunction).
Normalization: Due to the presence of ∂_λ derivatives + sufficiently high power of λ in the coefficients of the vector fields, the normalization contains not only the independent variables and λ, as it is in the classical dressing, but also the dependent variable.
Therefore the RH dressing allows one to obtain a spectral representation of the generic solution $u(x, y, t)$ of the PDE depending parametrically not only on x, y, t, but also sometimes on u itself;
Example: dKP $(u_t + uu_x)_x + u_{yy} = 0$: (MS 07)

$$\begin{aligned} &u = F\,(x - 2ut, y, t)\,, \\ &F\,(\xi, y, t) = -\int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R_2\Big(\pi_1^-(\lambda; \xi, y, t), \pi_2^-(\lambda; \xi, y, t)\Big). \end{aligned} \tag{14}$$

F is the spectral transform of the initial data.

Recall the simpler mechanism for the
Hopf equation $u_t + uu_x = 0$:
$u = F\,(x - 2ut)$, $F(x) = u(x, 0)$ gradient catastrophe of 1D waves

Vector nonlinear RH dressing for 2ddToda (MS 09)

Consider the following vector nonlinear RH problem:

$$\xi_j^+(\lambda) = \xi_j^-(\lambda) + R_j(\xi_1^-(\lambda)+\nu_1(\lambda), \xi_2^-(\lambda)+\nu_2(\lambda)),$$
$$\lambda \in \Gamma, \quad j = 1,2 \tag{15}$$

on an arbitrary closed contour Γ of the complex λ plane including the origin, where $\vec{R}(\vec{s}) = (R_1(s_1,s_2), R_2(s_1,s_2))^T$ are given differentiable spectral data depending analytically on the second argument s_2 through $exp(is_2)$ and satisfying the constraint

$$\begin{array}{l} \{\mathcal{R}_1(s_1,s_2), \mathcal{R}_2(s_1,s_2)\}_{(s_1,s_2)} = 1, \\ \mathcal{R}_j(s_1,s_2) := s_j + R_j(s_1,s_2), \quad j = 1,2, \\ \{f,g\}_{(s_1,s_2)} := f_{s_1}g_{s_2} - f_{s_2}g_{s_1}; \end{array} \tag{16}$$

for the normalizations $\nu_j, \ j = 1,2$:

$$\vec{\nu} = \left(\begin{array}{c} \nu_1 \\ \nu_2 \end{array}\right) = \left(\begin{array}{c} (\zeta_1\lambda + \zeta_2\lambda^{-1})e^{\frac{\phi_t}{2}} - t - \zeta_1\phi_{\zeta_1} \\ i\ln\lambda + i\frac{\phi_t}{2} \end{array}\right) \tag{17}$$

and $\vec{\xi}^+ = (\xi_1^+, \xi_2^+)^T$ and $\vec{\xi}^- = (\xi_1^-, \xi_2^-)^T$ are the unknown vector solutions of the RH problem (40), analytic rispectively inside and outside the contour Γ and such that $\vec{\xi}^- \to \vec{0}$ as $\lambda \to \infty$. Then, assuming that the above RH problem and its linearized form are uniquely solvable, we have the following results.

1) If

$$\begin{array}{l} \lim_{\lambda\to\infty} (i\lambda\xi_2^-) = \phi_{\zeta_1} e^{-\frac{\phi_t}{2}}, \\ i\xi_2^+(0) = \phi_t, \end{array} \tag{18}$$

it follows that $\vec{\pi}^{\pm} = \vec{\xi}^{\pm} + \vec{\nu}$ are common eigenfunctions of $\widehat{L}_{1,2}$: $\widehat{L}_1\vec{\pi}^{\pm} = \widehat{L}_2\vec{\pi}^{\pm} = \vec{0}$ satisfying the relations

$$\{\pi_1^{\pm}, \pi_2^{\pm}\}_{(\lambda,t)} = \lambda(\pi_1^{\pm}{}_{\lambda}\pi_2^{\pm}{}_{t} - \pi_1^{\pm}{}_{t}\pi_2^{\pm}{}_{\lambda}) = i \tag{19}$$

and the potentials ϕ_{ζ_1}, ϕ_t, constructed through (18), solve the 2ddT equation

$$\phi_{\zeta_1\zeta_2} = \left(e^{\phi_t}\right)_t, \quad \phi = \phi(\zeta_1, \zeta_2, t), \tag{20}$$

2) In addition, if the variables ζ_1, ζ_2 are specified as

$$\zeta_1 = z = \frac{x+iy}{2}, \quad \zeta_2 = \bar{z} = \frac{x-iy}{2}, \quad x, y \in \mathbb{R}, \tag{21}$$

so that the 2ddToda equation takes the nonlinear wave form $\phi_{xx} + \phi_{yy} = \left(e^{\phi_t}\right)_t$, if the RH data satisfy the additional reality constraint

$$\vec{\mathcal{R}}\left(\overline{\vec{\mathcal{R}}(\overline{\vec{s}})}\right) = \vec{s}, \quad \forall \vec{s} \in \mathbb{C} \tag{22}$$

and Γ is the unit circle, then the eigenfunctions satisfy the following symmetry relation:

$$\vec{\pi}^-(\lambda) = \overline{\vec{\pi}^+(1/\bar{\lambda})} \tag{23}$$

and $\phi \in \mathbb{R}$.

Since the normalization of the RH problem depends on the solution of 2ddT, the breaking mechanism of dKP is present also for the 2ddToda equation (in a more complicated form).

IST and Cauchy problem for the nonlinear wave equ.

Since the Lax pair is made of Hamiltonian vector fields:
1) The space of eigenfunctions is a ring: if $f_1,\ f_2$ are two eigenfunctions, then an arbitrary differentiable function $F(f_1, f_2)$ of them is also an eigenfunction.
2) the space of eigenfunctions is also a Lie algebra, whose Lie bracket is the natural Poisson bracket: if $f_1,\ f_2$ are two eigenfunctions, then their Poisson bracket $\{f_1, f_2\}$ is also an eigenfunction.

Cauchy problem for rapidly decreasing data:

$$\begin{array}{l}\left(e^{\phi_t}\right)_t = \phi_{xx} + \phi_{yy}, \quad x, y \in \mathbb{R}, \quad t > 0, \quad \phi(x, y, t) \in \mathbb{R}, \\ \phi(x, y, 0) = A(x, y), \quad \phi_t(x, y, 0) = B(x, y). \\ A(x, y), B(x, y) \to\ 0, \quad (x^2 + y^2) \to \infty \end{array} \tag{24}$$

Adding and subtracting the TT Lax pair, one obtains

$$\widehat{\mathcal{L}}_1\psi := \lambda\psi_{\bar{z}} - \lambda^{-1}\psi_z - \Big(- 2v_t + \lambda\tfrac{\phi_{\bar{z}t}}{2} + \lambda^{-1}\tfrac{\phi_{zt}}{2}\Big)\lambda\psi_\lambda = 0, \tag{25}$$

$$\widehat{\mathcal{L}}_2\psi := \psi_t + \tfrac{v^{-1}}{2}\left(\lambda\psi_{\bar{z}} + \lambda^{-1}\psi_z\right) - \tfrac{v^{-1}}{4}\left(\lambda\phi_{\bar{z}t} - \lambda^{-1}\phi_{zt}\right)\lambda\psi_\lambda = 0, \tag{26}$$

where the first equation must be viewed as the spectral problem and the second equation as t-evolution of the eigenfunction. Since the undressed operator: $\lambda\partial_{\bar{z}} - \lambda^{-1}\partial_z$ coincides with that of the spectral problem for the $(2+1)$-dimensional SDYM equation [MZ], the construction of the Jost and analytic Green's functions is taken from there.

Jost eigenfunctions and scattering vector are constructed on the unit circle of the complex λ plane, where $\lambda = exp(-i\theta)$. Introducing the convenient real variables ξ, η, θ':

$$\begin{aligned} &\xi = \cos\theta\ x + \sin\theta\ y, \\ &\eta = -\sin\theta\ x + \cos\theta\ y, \\ &\theta' = \theta, \end{aligned} \tag{27}$$

the Lax pair (25),(26) becomes

$$\hat{\mathcal{L}}_1\psi := \psi_\eta - \frac{1}{2}\left[-\left(\phi_{\xi\xi} + \phi_{\eta\eta}\right)v^{-1} + \phi_{\xi t}\right](\eta\psi_\xi - \xi\psi_\eta + \psi_{\theta'}) = 0, \tag{28}$$

$$\hat{\mathcal{L}}_2\psi := \psi_t + v^{-1}\psi_\xi - (v^{-1})_\eta(\eta\psi_\xi - \xi\psi_\eta + \psi_{\theta'}) = 0. \tag{29}$$

A convenient basis of Jost eigenfunctions are the solutions J_1 and J_2 satisfying the boundary conditions

$$\vec{J}(\xi,\eta,\theta') := \begin{pmatrix} J_1(\xi,\eta,\theta') \\ J_2(\xi,\eta,\theta') \end{pmatrix} \to \begin{pmatrix} \xi \\ \theta' \end{pmatrix}, \quad \text{as } \eta \to -\infty. \tag{30}$$

$\vec{J}$ solves the linear integral equation

$$\vec{J} = \begin{pmatrix} \xi \\ \theta' \end{pmatrix} + \frac{1}{2}\int_{-\infty}^{\eta} d\eta'\Big(-(\phi_{\xi\xi} + \phi_{\eta'\eta'})v^{-1} + \phi_{\xi t}\Big)(\eta'\vec{J}_\xi - \xi\vec{J}_{\eta'} + \vec{J}_{\theta'}). \tag{31}$$

The $\eta \to \infty$ limit of $\vec{f}$ defines the scattering vector $\vec{\sigma}(\xi,\theta) = (\sigma_1(\xi,\theta), \sigma_2(\xi,\theta))^T$

$$\vec{J}(\xi,\eta,\theta) \to \vec{\mathcal{S}}(\xi,\theta) = \begin{pmatrix} \xi \\ \theta \end{pmatrix} + \vec{\sigma}(\xi,\theta), \quad \text{as} \quad \eta \to \infty; \tag{32}$$

namely:

$$\vec{\sigma}(\xi,\theta) = \frac{1}{2}\int\limits_{\mathbb{R}} d\eta \left[-(\phi_{\xi\xi} + \phi_{\eta\eta})v^{-1} + \phi_{\xi t}\right](\eta\vec{J}_{\xi} - \xi\vec{J}_{\eta} + \vec{J}_{\theta}). \tag{33}$$

$J_1(\xi,\eta,\theta)$, $(J_2(\xi,\eta,\theta) - \theta)$ and $\vec{\sigma}(\xi,\theta)$ are 2π-periodic in θ (their dependence on the second argument θ is through $exp(i\theta)$).

Analytic eigenfunctions $\vec{\pi}^{\pm}$:

$$\vec{\pi}^{\pm}(z,\bar{z},\lambda) = \left(\begin{array}{c} \pi_1^{\pm}(z,\bar{z},\lambda) \\ \pi_2^{\pm}(z,\bar{z},\lambda) \end{array} \right) = \left(\begin{array}{c} \lambda z + \lambda^{-1}\bar{z} - t \\ i \ln \lambda \end{array} \right) + \frac{i}{4}\int_{\mathbb{C}} dz' \wedge d\bar{z}' G^{\pm}(z-z',\bar{z}-\bar{z}',\lambda)\Big(-\phi_{z\bar{z}}v^{-1} + \lambda\frac{\phi_{\bar{z}'t}}{2} + \lambda^{-1}\frac{\phi_{z't}}{2}\Big)\lambda\vec{\pi}^{\pm}_{\lambda}(z',\bar{z}',\lambda), \tag{34}$$

where $G^{\pm}$ are the Green's functions ($\lambda G^{\pm}_{\bar{z}} - \lambda^{-1}G^{\pm}_{z} = \delta(z)$):

$$G^{\pm}(z,\bar{z},\lambda) = \mp\frac{1}{\pi}\frac{1}{(\lambda z + \lambda^{-1}\bar{z})}, \quad \mathrm{sgn}(1-|\lambda|) = \pm 1 \tag{35}$$

analytic inside (+) and outside (-) the unit circle.
Since, on the unit circle $|\lambda| = 1$:

$$\begin{array}{ll} G^{+}(z-z',\bar{z}-\bar{z}',\lambda) \to -\frac{1}{\pi}\frac{1}{\xi-\xi'\pm i\varepsilon}, & \text{as } \eta \to \mp\infty, \\ G^{-}(z-z',\bar{z}-\bar{z}',\lambda) \to \frac{1}{\pi}\frac{1}{\xi-\xi'\mp i\varepsilon}, & \text{as } \eta \to \mp\infty. \end{array} \tag{36}$$

the $\eta \to -\infty$ limit of (π_1^{+},π_2^{+}) and (π_1^{-},π_2^{-}) are analytic respectively in the upper and lower parts of the complex ξ plane, while the $\eta \to \infty$ limit of (π_1^{+},π_2^{+}) and (π_1^{-},π_2^{-}) are analytic respectively in the lower and upper parts of the complex ξ plane. This mechanism, first observed in [MZ], plays an important role in the IST for vector fields [MS].

Connecting both eigenfunctions.
Since the Jost eigenfunctions $\vec{J} = (J_1, J_2)^T$ are a good basis in the space of eigenfunctions of the Lax pair for $|\lambda| = 1$, one can express the analytic eigenfunctions in terms of them through the following formulae, valid for $|\lambda| = 1$:

$$\vec{\pi}^{\pm} = \vec{\mathcal{K}}^{\pm}(\vec{J}) = \vec{J} + \vec{\chi}^{\pm}(J_1, J_2) \tag{37}$$

defining the spectral data $\vec{\chi}^{\pm}$ as differentiable functions of two arguments.

Using the analiticity properties in ξ and the 2π-periodicity in θ, we obtain the linear integral equations connecting the (Fourier transforms of the) scattering data $\vec{\sigma}$ to the (Fourier transforms of the) spectral data $\vec{\chi}^{\pm}$:

$$\begin{gathered}\tilde{\vec{\chi}}^{\pm}(\omega, n) + \\ H(\pm\omega)\left(\tilde{\vec{\sigma}}(\omega, n) + \int_{\mathbb{R}} d\omega' \sum_{n'=-\infty}^{\infty} \tilde{\vec{\chi}}^{\pm}(\omega', n') Q(\omega', n', \omega, n)\right) = \vec{0},\end{gathered} \tag{38}$$

where H is the Heaviside step function and

$$\begin{aligned} Q(\omega', n', \omega, n) &= \int_{\mathbb{R}} \frac{d\xi}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(\xi(\omega'-\omega)+(n'-n)\theta)} \left(e^{i(\omega'\sigma_1(\xi,\theta)+n'\sigma_2(\xi,\theta))} - 1\right), \\ \tilde{\vec{\chi}}^{\pm}(\omega, n) &= \int_{\mathbb{R}} d\xi \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i(\omega\xi+n\theta)} \vec{\chi}^{\pm}(\xi, \theta), \\ \tilde{\vec{\sigma}}(\omega, n) &= \int_{\mathbb{R}} d\xi \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i(\omega\xi+n\theta)} \vec{\sigma}(\xi, \theta). \end{aligned} \tag{39}$$

At last, eliminating, from equations (37), the Jost eigenfunctions $\vec{J}$, one obtains, through algebraic manipulation, the vector nonlinear RH problem on the unit circle we started with:

$$\begin{array}{l} \xi_j^+(\lambda) = \xi_j^-(\lambda) + R_j(\xi_1^-(\lambda) + \nu_1(\lambda), \xi_2^-(\lambda) + \nu_2(\lambda)), \\ \lambda \in \Gamma, \quad j = 1, 2, \\ \vec{\pi}^\pm = \vec{\xi}^\pm + \vec{\nu} \end{array} \tag{40}$$

t-evolution of the data

We remark that, as ϕ evolves according to the 2ddToda equation, the time evolution of the data is elementary:

$$\begin{array}{l} \vec{\sigma}(\xi,\theta,t) = \vec{\sigma}(\xi - t,\theta,0), \quad \vec{\chi}^\pm(\xi,\theta,t) = \vec{\chi}^\pm(\xi - t,\theta,0), \\ \vec{R}(\xi,\theta,t) = \vec{R}(\xi - t,\theta,0). \end{array} \tag{41}$$

Small field limit and Radon Transform

As for the IST of the heavenly and dKP equations, in the small field limit $|\phi|, |\phi_t| << 1$, the direct and inverse spectral transforms reduce to the direct and inverse Radon transforms.
The Direct problem: $\{A(x,y), B(x,y)\} \quad \rightarrow \quad \vec{\sigma}$ reduces to the direct Radon transform:

$$\begin{array}{l} \vec{\sigma}(\xi,\theta) \sim \frac{1}{2} \int\limits_{\mathbb{R}} \left(\begin{array}{c} \eta \\ 1 \end{array} \right) \Big[-(\partial_\xi^2 + \partial_\eta^2) A(x(\xi,\eta,\theta), y(\xi,\eta,\theta)) + \\ \partial_\xi B(x(\xi,\eta,\theta), y(\xi,\eta,\theta)) \Big] d\eta, \\ x(\xi,\eta,\theta) = \xi \cos\theta - \eta \sin\theta, \quad y(\xi,\eta,\theta) = \xi \sin\theta + \eta \cos\theta, \end{array} \tag{42}$$

while the spectral data $\vec{\chi}^{\pm}$ and $\vec{R}$ are constructed from $\vec{\sigma}$ as follows:

$$\vec{\chi}^{\pm}(\xi,\theta) \sim -\widehat{P}_{\xi}^{\pm}\vec{\sigma}(\xi,\theta), \quad \vec{R}(\xi,\theta) \sim -i\widehat{\mathcal{H}}_{\xi}\vec{\sigma}(\xi,\theta), \tag{43}$$

where $\widehat{P}_{\xi}^{\pm}$ and $\widehat{\mathcal{H}}_{\xi}$ are rispectively the $(\pm)$ analyticity projectors and the Hilbert transform in the variable ξ.
At last, the first of the closure conditions of the inverse problem reduces to the inverse Radon transform

$$\phi_t(x,y,t) \sim -\frac{1}{2\pi i}\int_0^{2\pi} d\theta R_2(\xi - t,\theta) \sim -\frac{1}{2\pi^2}\int_0^{2\pi} d\theta \;\; P\int_{\mathbb{R}} \frac{d\xi'}{\xi'-(\xi-t)}\sigma_2(\xi',\theta),$$
$$\xi = x\cos\theta + y\sin\theta, \tag{44}$$

that can be shown to be equivalent to the well-known Poisson formula

$$\phi(x,y,t) = \partial_t \int_{\mathbb{R}^2} \frac{dx'dy'}{2\pi} L(x-x',y-y',t)A(x',y') + \int_{\mathbb{R}^2} \frac{dx'dy'}{2\pi} L(x-x',y-y',t)B(x',y'), \tag{45}$$

where

$$L(x,y,t) := \frac{H(t^2-x^2-y^2)}{\sqrt{t^2-x^2-y^2}} \tag{46}$$

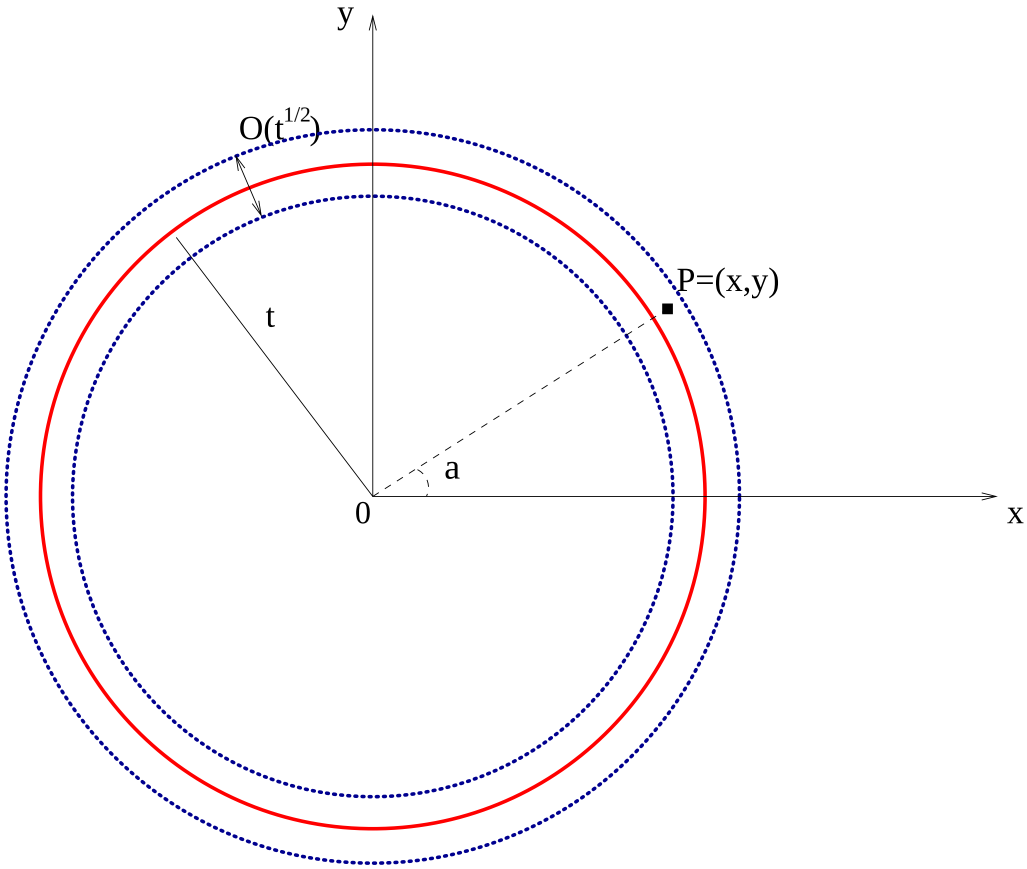
and $H(\cdot)$ is the Heaviside step function, describing the solution of the Cauchy problem

$$\phi_{tt} = \phi_{xx} + \phi_{yy}, \quad x,y \in \mathbb{R}, \quad t > 0, \quad \phi(x,y,t) \in \mathbb{R},$$
$$\phi(x,y,0) = A(x,y), \quad \phi_t(x,y,0) = B(x,y). \tag{47}$$

for the linear wave equation in 2+1 dimensions.

Longtime behaviour of 2ddT solutions,for small data, in the space-time region

$$\begin{array}{l} z = \frac{t+r}{2}e^{i\alpha}, \qquad \alpha, r \in \mathbb{R}, \qquad t >> 1, \\ X := \sqrt{x^2+y^2} - t + \sqrt{x^2+y^2}\phi_t + \frac{3t}{8}\phi_t^2 = O(1), \\ \alpha = \arctan(y/x) = O(1), \\ r = \sqrt{x^2+y^2} - t, \end{array} \qquad (48)$$



the behaviour of the solutions of the 2ddT equation $(exp\phi_t)_t = \phi_{xx} + \phi_{yy}$ (for small initial data) is described by the following implicit (scalar) equation:

$$\phi_t = \frac{1}{\sqrt{t}}F(X,\alpha) + o\left(\frac{1}{\sqrt{t}}\right), \qquad (49)$$

where F is given by

$$F\,(X,\alpha) = -\frac{1}{2\pi i}\int\limits_{\mathbb{R}} d\mu' R_2\Big(-\frac{\mu'^2}{2} + X + a_1(\mu';X,\alpha),\\ \alpha + a_2(\mu';X,\alpha)\Big) \tag{50}$$

and $a_j(\mu;X,\alpha),\ j=1,2$ are the solutions of the integral equations

$$a_j(\mu;X,\alpha) = \frac{1}{2\pi i}\int\limits_{\mathbb{R}} \frac{d\mu'}{\mu'-(\mu-i\epsilon)} R_j\Big(-\frac{\mu'^2}{2} + X + a_1(\mu';X,\alpha), \alpha +\\ a_2(\mu';X,\alpha)\Big), \quad j=1,2. \tag{51}$$

Outside the asymptotic region (48) the solution decays faster.

Therefore the 2ddT solutions break also in the longtime regime, and this breaking is essentially described by the universality features of the dKP theory.

COMPARISON WITH dKP

dKP: Vector nonlinear RH dressing on the real line (MS 06)

$$\begin{aligned} &\vec{\pi}^{+}(\lambda) = \vec{\pi}^{-}(\lambda) + \vec{R}(\vec{\pi}^{-}(\lambda)), \quad \lambda \in \mathbb{R}, \\ &\vec{\pi}^{\pm}(\lambda) = \vec{\nu}(\lambda; x-2ut, y, t) + \vec{O}(\lambda^{-1}), \end{aligned} \tag{52}$$

where

$$\begin{aligned} &\vec{\nu}(\lambda; x-2ut, y, t) = \begin{pmatrix} -\lambda^2 t - \lambda y + x - 2ut \\ \lambda \end{pmatrix}, \\ &u = \lim_{\lambda\to\infty} \big(\lambda(\pi_2^{\pm}(\lambda) - \lambda\big), \end{aligned} \tag{53}$$

and the spectral data $\vec{R}(\vec{\zeta}) = (R_1(\zeta_1,\zeta_2), R_2(\zeta_1,\zeta_2)) \in \mathbb{C}^2$, $\vec{\zeta} \in \mathbb{C}^2$, satisfy the following properties:

$$\begin{aligned} &\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\overline{\vec{\zeta}})}) = \vec{\zeta}, \quad \forall \vec{\zeta} \in \mathbb{C}^2, && \text{reality} \\ &R_{1\zeta_1} + R_{2\zeta_2} + \{R_1, R_2\}_{\vec{\zeta}} = 0, && \text{dKP constraint.} \end{aligned} \tag{54}$$

Then $u = F(x-2ut, y, t) \in \mathbb{R}$ is solution of the dKP equation, where

$$F(\xi, y, t) = -\int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R_2\Big(\pi_1^{-}(\lambda; \xi, y, t), \pi_2^{-}(\lambda; \xi, y, t)\Big). \tag{55}$$

The solution of this Riemann problem depends parametrically on $(x-2ut, y, t)$ through the normalization at ∞. The inverse formula is an implicit equation for the dKP solution, similar to the solution of the 1+1 dimensional Hopf equation.

The longtime behavior of dKP solutions

Let $t >> 1$ and

$$\begin{array}{l} x = \tilde{x} + v_1 t, \quad y = v_2 t, \\ \tilde{x} - 2ut, v_1, v_2 = O(1), \quad v_2 \neq 0, \quad t >> 1. \end{array} \tag{56}$$

On the parabola

$$x + \frac{y^2}{4t} = \tilde{x} \quad (v_1 = -\frac{v_2^2}{4}), \tag{57}$$

the longtime behaviour of the solutions of the dKP equation is given by

$$\begin{array}{l} u = \frac{1}{\sqrt{t}} G\Big(x - 2ut + \frac{y^2}{4t}, \frac{y}{2t}\Big) + o\left(\frac{1}{\sqrt{t}}\right), \\ G(\xi, \eta) = -\frac{1}{2\pi i} \int\limits_{\mathbb{R}} d\mu R_2\Big(\xi + \mu^2 + a_1(\mu; \xi, \eta), \eta + a_2(\mu; \xi, \eta)\Big), \end{array} \tag{58}$$

where $a_j(\mu : \xi, \eta)$ are associated with the following "asymptotic" vector nonlinear Riemann problem on the real axis:

$$\begin{array}{l} \vec{A}^+(\mu; \xi, \eta) = \vec{A}^-(\mu; \xi, \eta) + \vec{R}(\vec{A}^-(\mu; \xi, \eta)), \quad \mu \in \mathbb{R}, \\ \vec{A}^\pm(\mu; \xi, \eta) = \begin{pmatrix} \xi + \mu^2 \\ \eta \end{pmatrix} + \vec{a}(\mu; \xi, \eta). \end{array} \tag{59}$$

Outside the parabola, the solution decays faster. Small initial data begin evolving according to $u_{tx} + u_{yy} = 0$. Only in the longtime regime the nonlinear term becomes relevant, causing the breaking of the small localized initial wave in a point of the parabola.

The above asymptotics suggests the convenient variables:

$$V = \sqrt{t}\, u, \quad \tilde{x} = x + \frac{y^2}{4t}, \quad \tilde{y} = \frac{y}{2t}, \quad \tilde{t} = 2\sqrt{t}, \tag{60}$$

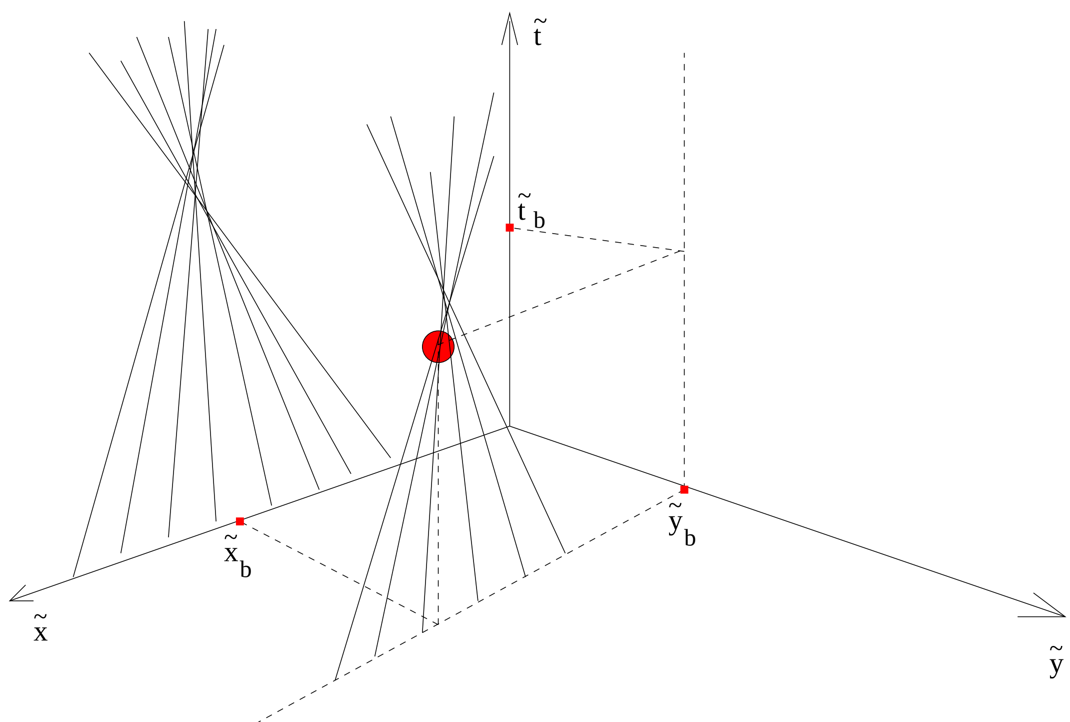
$\Rightarrow V$ evolves according to the $1+1$ dimensional Hopf equation:

$$V_{\tilde{t}} + V V_{\tilde{x}} = 0. \tag{61}$$

The longtime behavior of the dKP solutions is reduced to the study of the evolution of a two-dimensional localized wave under the $1+1$ dimensional Hopf equation (61).
Its solution is defined implicitely by the equations

$$V = G(\xi, \tilde{y}), \quad \tilde{x} = \xi + G(\xi, \tilde{y})\tilde{t}, \tag{62}$$

describing a 2-parameter family (with parameters $\xi, \tilde{y}$) of straight line characteristics.



Since

$$\nabla_{(\tilde{x},\tilde{y})} V = \frac{\nabla_{(\xi,\tilde{y})} G(\xi,\tilde{y})}{1+G_\xi(\xi,\tilde{y})\tilde{t}}, \tag{63}$$

the gradient catastrophe takes place on the two dimensional singularity manifold (SM):

$$\mathcal{S}(\xi,\tilde{y},t) \equiv 1 + G_\xi(\xi,\tilde{y})\tilde{t} = 0 \quad \Rightarrow \quad \tilde{t} = -\frac{1}{G_\xi(\xi,\tilde{y})}. \tag{64}$$

The first breaking space-time point $(\tilde{\vec{x}}_b, \tilde{y}_b, \tilde{t}_b)$ and the corresponding characteristic parameters $\vec{\xi}_b = (\xi_b, \tilde{y}_b)$: are defined by

$$\tilde{t}_b = -\frac{1}{G_{\tilde{\xi}}(\vec{\xi}_b)} = \text{global min}\left(-\frac{1}{G_\xi(\xi,\tilde{y})}\right) > 0, \tag{65}$$

,

$$\tilde{x}_b = \xi_b + G(\vec{\xi}_b)\tilde{t}_b. \tag{66}$$

$$\begin{array}{ll} G_\xi(\vec{\xi}_b) < 0, & G_{\xi\xi}(\vec{\xi}_b) = G_{\xi\tilde{y}}(\vec{\xi}_b) = 0, \\ G_{\xi\xi\xi}(\vec{\xi}_b) > 0, & \alpha \equiv G_{\xi\xi\xi}(\vec{\xi}_b) G_{\xi\tilde{y}\tilde{y}}(\vec{\xi}_b) - G^2_{\xi\xi\tilde{y}}(\vec{\xi}_b) > 0. \end{array} \tag{67}$$

Evaluating equations (62b) and (64) near breaking:

$$\begin{array}{l} \tilde{x} = \tilde{x}_b + \tilde{x}', \quad \tilde{y} = \tilde{y}_b + \tilde{y}', \quad \tilde{t} = \tilde{t}_b + \tilde{t}', \quad \xi = \xi_b + \xi', \\ |\tilde{x}'|, |\tilde{y}'|, |\tilde{t}'|, |\xi'| << 1, \end{array} \tag{68}$$

where $\tilde{x}', \tilde{y}', \tilde{t}', \xi'$ are small, at the leading order we get a cubic equation in ξ':

$$\xi'^3 + a(\tilde{y}')\xi'^2 + b(\tilde{y}',\tilde{t}')\xi' - \gamma X(\tilde{x}',\tilde{y}',\tilde{t}') = 0, \tag{69}$$

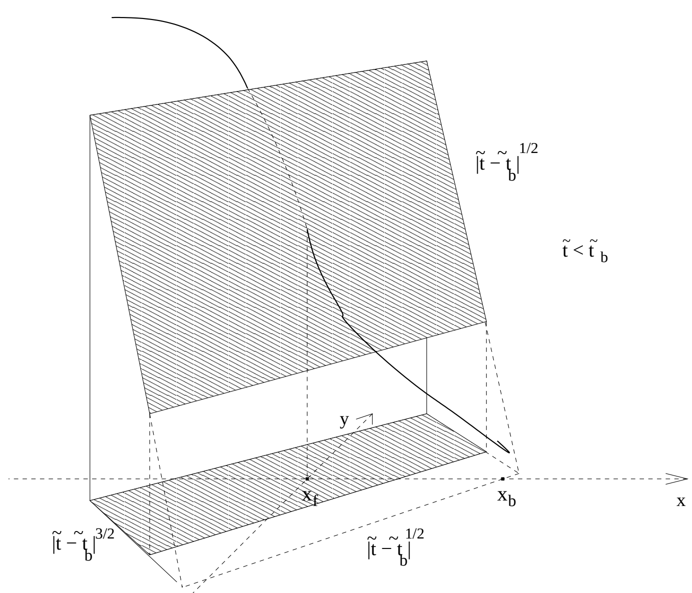
where

$$a(\tilde{y}') = \frac{3G_{\xi\xi\tilde{y}}}{G_{\xi\xi\xi}}\tilde{y}', \qquad b(\tilde{y}',\tilde{t}') = \frac{3}{G_{\xi\xi\xi}}\left[G_\xi \epsilon + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2\right],$$
$$X(\tilde{x}',\tilde{y}',\tilde{t}') = \tilde{x}' - G(\xi_b,\tilde{y}_b+\tilde{y}')\tilde{t}' - [G(\xi_b,\tilde{y}_b+\tilde{y}') - G]\,\tilde{t}_b \sim$$
$$\tilde{x}' + \frac{G_{\tilde{y}}}{G_\xi}\tilde{y}' - G\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}}}{2G_\xi}\tilde{y}'^2 - G_{\tilde{y}}\tilde{y}'\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_\xi}\tilde{y}'^3, \qquad \gamma = \frac{6|G_\xi|}{G_{\xi\xi\xi}}, \tag{70}$$

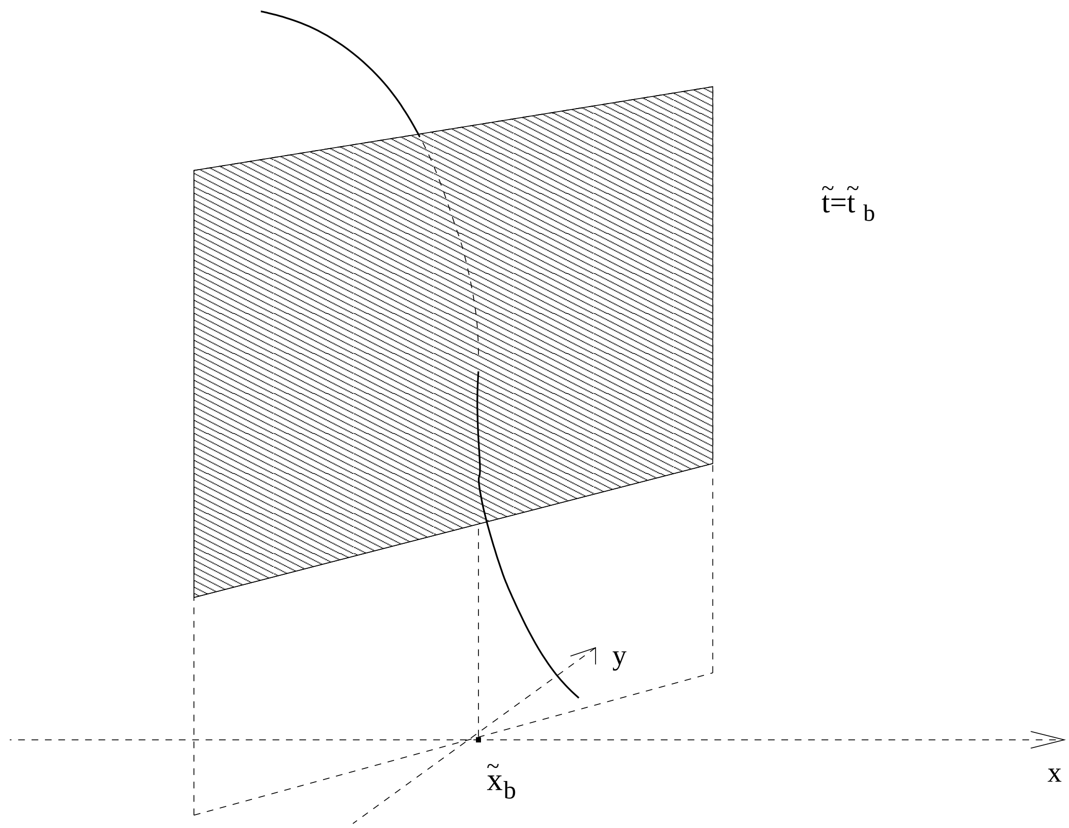
with the small parameter $\epsilon \equiv 2\frac{\tilde{t}-\tilde{t}_b}{\tilde{t}_b}$, solved by Cardano-Tartaglia formula.

Before breaking: $\tilde{t} < \tilde{t}_b$ and $\Delta > 0$ (single-valued solution). In a narrow region around the inflection point, the cubic becomes linear and the solution coincides with the exact similarity solution of the Hopf equation, describing the plane tangent to the wave at the inflection point:

$$V \sim \frac{\tilde{x}-\tilde{x}_b+(G_{\tilde{y}}/G_\xi)(\tilde{y}-\tilde{y}_b)}{\tilde{t}-\tilde{t}_b}.$$
$$\nabla_{(\tilde{x},\tilde{y})}V \sim \frac{1}{\tilde{t}'}\left(1,\frac{G_{\tilde{y}}}{G_\xi}\right), \tag{71}$$



At breaking: $\tilde{t} \uparrow \tilde{t}_b$, the inflection point becomes the breaking point: $\vec{x}_f \to (\tilde{x}_b, \tilde{y}_b)$, the above tangent plane becomes vertical, with equation $x' - G_{\tilde{y}}(\tilde{\xi}_b, \tilde{y}_b)y' = 0$, the above strip reduces to the breaking point $(\tilde{x}_b, \tilde{y}_b)$.

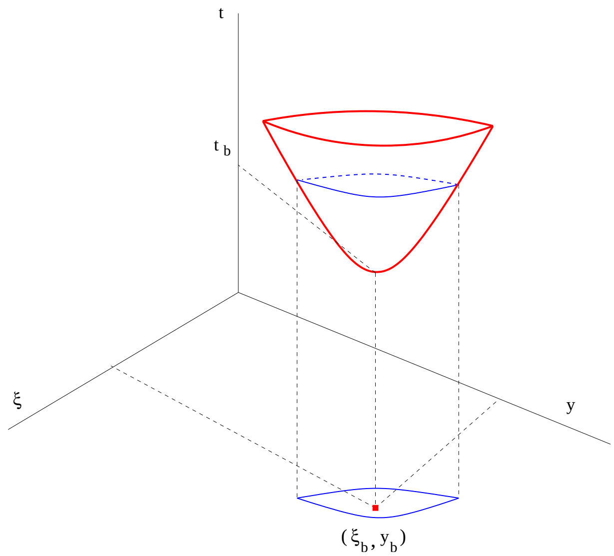


$$V \sim G\left(\xi_b + \sqrt[3]{\gamma(\tilde{x} - \tilde{x}_b)}, \tilde{y}\right) \quad \Rightarrow \quad V_{\tilde{x}} \sim \frac{\sqrt[3]{\gamma}}{3} \frac{G_\xi(\vec{\xi}_b)}{\sqrt[3]{(\tilde{x} - \tilde{x}_b)^2}}. \quad \tilde{y} = \tilde{y}_b,$$
$$V \sim G\left(\xi_b - \sqrt[3]{\frac{6G_{\tilde{y}}}{G_{\xi\xi\xi}}(\tilde{y} - \tilde{y}_b)}, \tilde{y}\right) \Rightarrow V_{\tilde{y}} \sim -\sqrt[3]{\frac{2G_{\tilde{y}}}{3G_{\xi\xi\xi}}} \frac{G_\xi(\vec{\xi}_b)}{\sqrt[3]{(\tilde{y} - \tilde{y}_b)^2}}. \; \tilde{x} = \tilde{x}_b \tag{72}$$

After breaking. If $\tilde{t} > \tilde{t}_b$ ($\tilde{t}' > 0$), the SM equation $\mathcal{S} = 0$:

$$G_{\xi\xi\xi}\xi'^2 + 2G_{\xi\xi\tilde{y}}\xi'\tilde{y}' + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2 = |G_\xi|\epsilon \tag{73}$$

describes an elliptic paraboloid in the $(\xi, \tilde{y}, \tilde{t})$ space, with minimum at the point $(\vec{\xi}_b, \tilde{t}_b)$

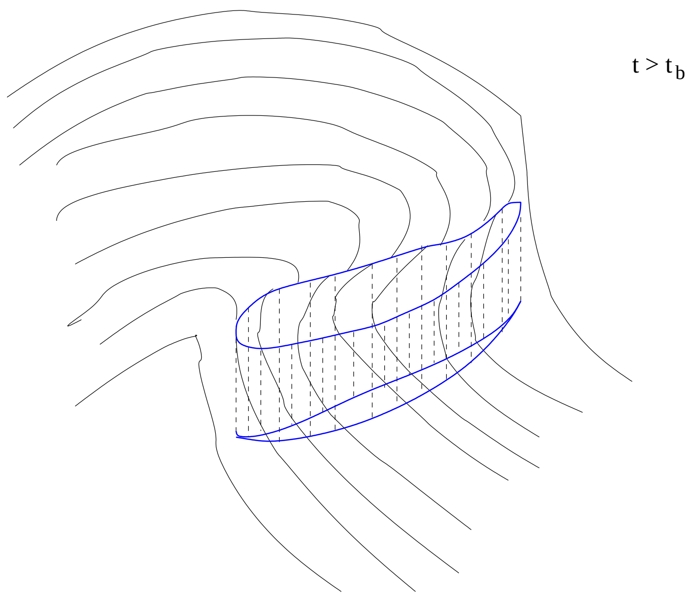


The SM equation in space-time coordinates:

$$\Big[3|G_\xi|G_{\xi\xi\xi}^2\left(\tilde{x}' + \frac{G_{\tilde{y}}}{G_\xi}\tilde{y}' - G\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}}}{2G_\xi}\tilde{y}'^2 - G_{\tilde{y}}\tilde{y}'\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_\xi}\tilde{y}'^3\right) + \frac{G_{\xi\xi\xi}G_{\xi\xi\tilde{y}}G_{\xi\tilde{y}\tilde{y}}}{2}\left(\frac{G_\xi}{G_{\xi\tilde{y}\tilde{y}}}\epsilon + y'^2\right)\tilde{y}' - \alpha G_{\xi\xi\tilde{y}}\left(\frac{|G_\xi|G_{\xi\xi\xi}}{\alpha}\epsilon - \tilde{y}'^2\right)\tilde{y}'\Big]^2 = \alpha^3\left(\frac{|G_\xi|G_{\xi\xi\xi}}{\alpha}\epsilon - \tilde{y}'^2\right)^3, \qquad \Delta = 0 \ \textit{condition} \tag{74}$$

a closed caustic of the $(\tilde{x}, \tilde{y})$ plane possessing the two cusps

$$\vec{\tilde{x}}_c^{\pm}(\tilde{t}') \sim \vec{\tilde{x}}_b \mp \sqrt{\frac{|G_\xi|G_{\xi\xi\xi}\epsilon}{\alpha}}\left(\frac{G_{\tilde{y}}}{G_\xi}, 1\right). \tag{75}$$



On the caustic $\Delta = 0$, the cubic has three real solutions, but two of the branches coincide and their slopes are ∞. At the cusps, all the three branches coincide

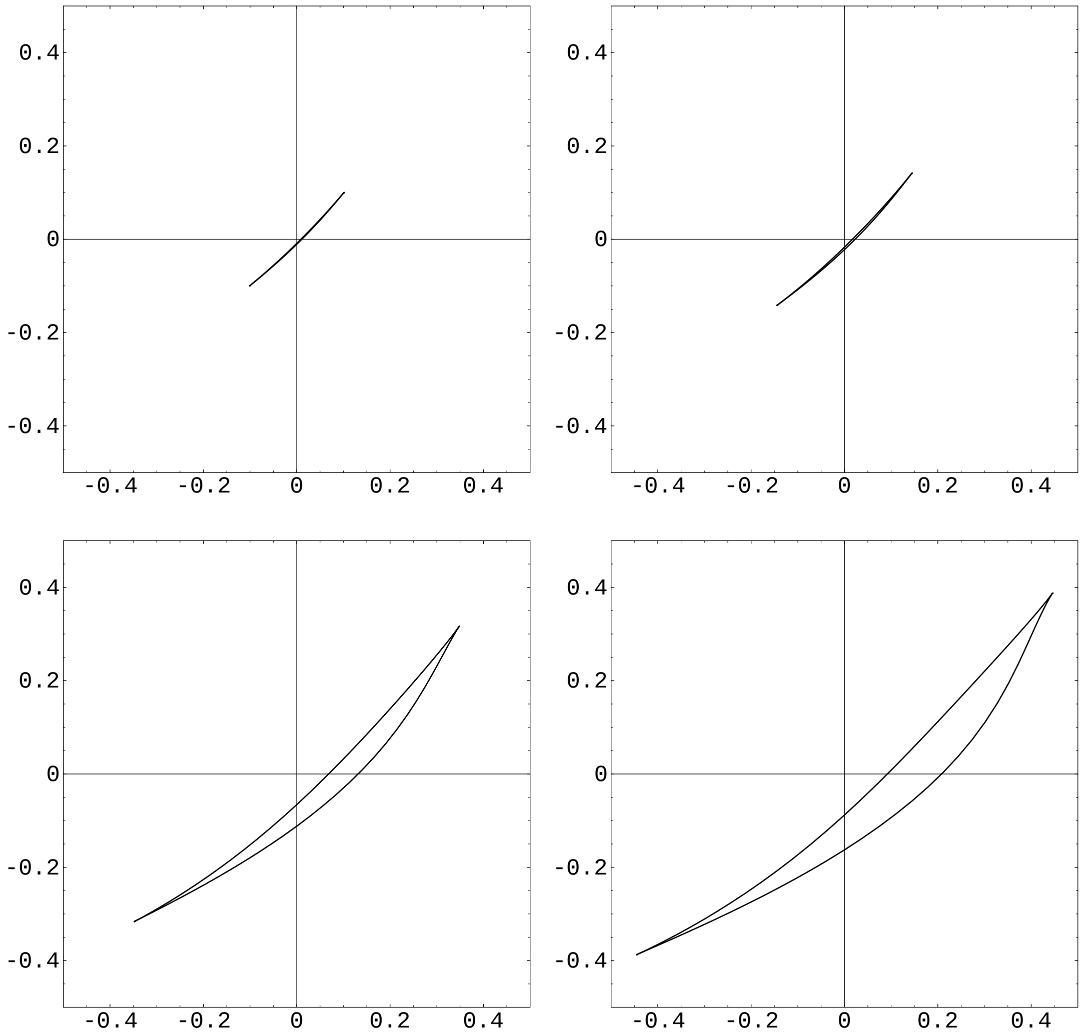
Inside the caustic, $\Delta < 0$ and the solution is three-valued. This multivalued region has to be replaced by a proper shock layer, whose features depend on the wanted regularization.

Outside the caustic, $\Delta > 0$ and the solution is single valued.

The closed caustic is the boundary of a narrow region of thickness $O(\epsilon^{3/2})$ in the longitudinal direction, and of thickness $O(\epsilon^{1/2})$ in the transversal direction (it develops with zero speed in the longitudinal direction, and with ∞ speed in the transversal direction).

Such an universal behaviour appears also in a simpler 2+1 dimensional model (E. Kuznetsov)

Four snapshots describing the evolution of such caustic immediately after breaking:



The similarity solution before breaking, the vertical inflection at breaking, and the caustic after breaking make clear the universal character of the gradient catastrophe of two-dimensional waves, for the Hopf and dKP equations.

Summarizing: the gradient catastrophe for dKP, in the longtime regime, is described by the solutions of the universal and explicit cubic equation

$$\xi'^3 + a(\tilde{y}')\xi'^2 + b(\tilde{y}', \tilde{t}')\xi' - \gamma X(\tilde{x}', \tilde{y}', \tilde{t}') = 0, \qquad (76)$$

where

$$a(\tilde{y}') = \frac{3G_{\xi\xi\tilde{y}}}{G_{\xi\xi\xi}}\tilde{y}', \qquad b(\tilde{y}', \tilde{t}') = \frac{3}{G_{\xi\xi\xi}}\left[G_{\xi}\epsilon + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2\right],$$
$$X(\tilde{x}', \tilde{y}', \tilde{t}') = \tilde{x}' - G(\xi_b, \tilde{y}_b + \tilde{y}')\tilde{t}' - [G(\xi_b, \tilde{y}_b + \tilde{y}') - G]\,\tilde{t}_b \sim$$
$$\tilde{x}' + \frac{G_{\tilde{y}}}{G_{\xi}}\tilde{y}' - G\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}}}{2G_{\xi}}\tilde{y}'^2 - G_{\tilde{y}}\tilde{y}'\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_{\xi}}\tilde{y}'^3, \qquad \gamma = \frac{6|G_{\xi}|}{G_{\xi\xi\xi}}, \qquad (77)$$

the dependence on the initial data appears just through the constant coefficients $G, G_{\xi}, \ldots$, where G is the spectral function and its derivatives evaluated at the breaking point.
$O(1)$ initial data break at finite t. Although the picture is more complicated and implicit, one can establish also in this case that: i) the first breaking takes place in a point of the (x, y, t) space; ii) the gradient catastrophe is ruled by a cubic equation; iii) the exact similarity solution of dKP describes the tangent plane to the 2D profile at the inflection point, immediately before breaking.

A distinguished class of implicit solutions

Suppose that the two components of the RH spectral data $\vec{R}$ in (40) are given by:

$$R_j(s_1, s_2) = (-1)^{j+1} i f\left(e^{s_1+s_2}\right), \quad j = 1, 2, \tag{78}$$

in terms of the single real spectral function f of a single argument, depending on s_1 and s_2 only through their sum. Then:

1) The reality and Hamiltonian constraints (22) and (16) are satisfied.

2)

$$\begin{aligned} w(\lambda) := \pi_1^+ + \pi_2^+ = \pi_1^- + \pi_2^- = (z\lambda + \bar{z}\lambda^{-1})v+ \\ i \ln \lambda - t - \bar{z}\phi_{\bar{z}} - i\frac{\phi_t}{2}, \end{aligned} \tag{79}$$

implying:

$$\phi_t = i(\bar{z}\phi_{\bar{z}} - z\phi_z) \tag{80}$$

and the following nonlinear two dimensional constraint:

$$i(\bar{z}\partial_{\bar{z}} - z\partial_z)\left(e^{i(\bar{z}\phi_{\bar{z}} - z\phi_z)}\right) = \phi_{z\bar{z}} \tag{81}$$

3) The RH problem linearizes

$$\begin{aligned} \pi_1^+ = \pi_1^- + i f\left(e^{w(\lambda)}\right), \\ \pi_2^+ = \pi_2^- - i f\left(e^{w(\lambda)}\right), \end{aligned} \tag{82}$$

and its explicit solution is:

$$\xi_j^{\pm}(\lambda) = (-1)^{j+1}\frac{1}{2\pi i}\oint_{|\lambda|=1} \frac{d\lambda'}{\lambda' - (1\mp\epsilon)e^{arg\lambda}} f\left(e^{w(\lambda')}\right), \quad j = 1, 2. \tag{83}$$

4) The closure conditions (18) read

$$\phi_z e^{-\frac{\phi_t}{2}} = -\frac{1}{2\pi i}\oint_{|\lambda|=1} d\lambda f\left(e^{w(\lambda)}\right), \;\; \phi_t = \frac{1}{2\pi i}\oint_{|\lambda|=1} \frac{d\lambda}{\lambda} f\left(e^{w(\lambda)}\right). \tag{84}$$

Although the RH problem (82) is linear, since $w(\lambda)$ depends on the unknowns ϕ_t, ϕ_z, the closure conditions (84) are a nonlinear algebraic system of two equations for the two unknowns ϕ_t, ϕ_z, defining implicitly a class of solutions of the 2ddT equation parametrized by the arbitrary real spectral function $f(\cdot)$ of a single variable.

Do all integrable dispersionless PDEs arising as commutation of vector fields feature the gradient catastrophe of localized initial data at finite time? NO

Example 1: heavenly equation $\theta_{zy} - \theta_{tx} + \theta_{xy}^2 - \theta_{xx}\theta_{yy} = 0$
Vector nonlinear RH dressing:
Consider the following nonlinear RH problem on the real λ-axis:

$$\vec{\pi}^+ = \vec{\pi}^- + \vec{R}(\vec{\pi}^-, \lambda) \equiv \vec{\mathcal{R}}(\vec{\pi}^-, \lambda), \quad \lambda \in \mathbb{R}, \tag{85}$$

for the functions $\vec{\pi}^+(\vec{x}, z, t, \lambda)$ and $\vec{\pi}^-(\vec{x}, z, t, \lambda)$, analytic respectively in the upper and lower halves of the complex plane λ, normalized as follows for $|\lambda| >> 1$:

$$\vec{\pi}^{\pm}(\vec{x}, z, \lambda) = \begin{pmatrix} x - \lambda z \\ y - \lambda t \end{pmatrix} - \frac{\vec{U}(\vec{x}, z, t)}{\lambda} + O(\lambda^{-2}). \quad \vec{x} = (x, y), \tag{86}$$

where the vector $\vec{R}(\vec{\pi}^-, \lambda)$ depends on $(\vec{x}, z, t)$ only through $\vec{\pi}^-$. Assume also the unique solvability of the RH problem (85) and of its linearized version

$$\begin{array}{l} \vec{\nu}^+ = \vec{\nu}^- + \rho(\vec{\pi}^-, \lambda)\vec{\nu}^-, \quad \lambda \in \mathbb{R}, \\ (\rho)_{ij}(\vec{\zeta}, \lambda) \equiv \frac{\partial R_i(\vec{\zeta}, \lambda)}{\partial \zeta_j}. \end{array} \tag{87}$$

Then $\vec{\pi}^{\pm}$ are solutions of $\hat{L}_1\vec{\pi}^{\pm} = \hat{L}_2\vec{\pi}^{\pm} = \vec{0}$, where $\hat{L}_1, \hat{L}_2$ are defined by:

$$\begin{array}{ll} \hat{L}_1 = \partial_z + \lambda\partial_x + \vec{u}_1 \cdot \nabla_{\vec{x}}, & \vec{u}_1 = \vec{U}_x, \\ \hat{L}_2 = \partial_t + \lambda\partial_y + \vec{u}_2 \cdot \nabla_{\vec{x}}, & \vec{u}_2 = \vec{U}_y, \end{array} \tag{88}$$

It follows that $\vec{U}$ solves the nonlinear system of PDEs

$$\begin{array}{l}\vec{U}_{tx} - \vec{U}_{zy} + \left(\vec{U}_y \cdot \nabla_{\vec{x}}\right) \vec{U}_x - \left(\vec{U}_x \cdot \nabla_{\vec{x}}\right) \vec{U}_y = \vec{0}, \\ \vec{U} \in \mathbb{R}^2, \quad \vec{x} = (x, y), \quad \nabla_{\vec{x}} = (\partial_x, \partial_y),\end{array} \tag{89}$$

If, in addition, the spectral data satisfy the Hamiltonian constraint

$$\{\mathcal{R}_1, \mathcal{R}_2\}_{\vec{\zeta}} = 1, \tag{90}$$

then $\vec{U} = (\theta_y, -\theta_x)$ and θ satisfies the heavenly equation. At last, the reality constraints

$$\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\overline{\vec{\zeta}}, \lambda)}, \lambda) = \vec{\zeta}, \quad \forall \vec{\zeta}, \quad \lambda \in \mathbb{R} \tag{91}$$

imply that $\overline{\vec{\pi}^{+}} = \vec{\pi}^{-}$, and the reality of θ.

Since the normalization is given in terms of x, y, z, t, λ only, there is no spectral mechanism for breaking!

Example 2: Pavlov equ: $v_{xt} + v_{yy} - v_y v_{xx} + v_x v_{xy} = 0$, $u = 0$ reduction of

$$\begin{aligned} &u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} = 0, \\ &v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} = 0, \end{aligned} \tag{92}$$

$v = 0$: dKP equ. $u_{xt} + u_{yy} + (uu_x)_x = 0$

They have the same linear term, but different nonlinearity ...

RH dressing:

Consider the scalar nonlinear RH problem on the real line:

$$\pi^+(\lambda) = \pi^-(\lambda) + R(\pi^-(\lambda), \lambda), \quad \lambda \in \mathbb{R}, \tag{93}$$

where $\pi^+(\lambda), \pi^-(\lambda) \in \mathbb{C}$ are scalar functions analytic in the upper and lower halves of the complex λ plane, normalized as follows

$$\begin{aligned} &\pi^\pm(\lambda; x, y, t) = \nu(\lambda; x, y, t) + O(\lambda^{-1}), \quad |\lambda| >> 1, \\ &\nu(\lambda; x, y, t) = -\lambda^2 t - \lambda y + x, \end{aligned} \tag{94}$$

and the spectral datum $R(\zeta, \lambda) \in \mathbb{C}$ satisfies the following reality constraint:

$$\mathcal{R}(\overline{\mathcal{R}(\bar{\zeta}, \lambda)}, \lambda) = \zeta, \quad \forall \zeta \in \mathbb{C}, \quad \lambda \in \mathbb{R}, \tag{95}$$

where $\mathcal{R}(\zeta, \lambda) := \zeta + R(\zeta, \lambda)$. Then, assuming uniqueness of the solution of such a RH problem and of its linearized version, it follows that $\pi^\pm$ are solutions of the linear problems $\hat{L}_{1,2}\pi^\pm = 0$, where

$$\begin{aligned} &\hat{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x, \\ &\hat{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x - v_y)\partial_x. \end{aligned} \tag{96}$$

and

$$v(x,y,t) = \int\limits_{\mathbb{R}} \frac{d\lambda}{2\pi i} R\Big(\pi^-(\lambda; x, y, t), \lambda\Big) \quad (97)$$

is solution of Pavlov equation.
Again the normalization of the RH problem does not depend on the dependent variable and the spectral mechanism for breaking is absent!

The common features of these two examples is the lack, in the vector fields, of the partial derivative wrt the spectral parameter...

==============================

References for this talk:

1) MS, Phys. Lett. A **359** (2006) (heavenly)
2) MS, JETP Letters **83** (2006) (dKP system + dKP)
3) MS, Theor. Math. Phys. **152** (2007) (Pavlov's equ.)
4) MS, J.Phys.A: Math.Theor. **41** (2008) 055204. (asymptotics, particular solutions and wave breaking in dKP)
5) MS, J. Phys. A: Math. Theor. **42** (2009) 095203 (16pp) (Cauchy problem, asymptotics, particular solutions and wave breaking for 2ddT).
6) MS, J. Phys. A: Math. Theor. 2009 (in press) (asymptotics and particular solutions for heavenly and Pavlov equation).