B.A. Dubrovin, A.Ya. Maltsev.

Deformations of Whitham systems and dispersive analog of shock waves.

1 Deformation scheems.

Let us consider the KdV equation

$$\varphi_t + \varphi\varphi_x + \varphi_{xxx} = 0 \qquad (1.1)$$

We are trying to find a solution of (1.1) having the form

$$\phi(\theta, X, T) = \Phi\left(S(X,T) + \theta, k, \omega, n\right) + \sum_{l \geq 1} \Phi_{(l)}\left(S(X,T) + \theta, X, T\right) \qquad (1.2)$$

where all $\Phi_{(l)}(\theta, X, T)$ are local expressions depending on $(k, \omega, n, k_X, \omega_X, n_X, \dots$ polynomial in derivatives $(k_X, \omega_X, n_X, \dots)$ and having gradation l according to the following definition

All the functions $f(k, \omega, n)$ have degree 0;
The derivatives k_{lX}, ω_{lX}, n_{lX} have degree l;
The degree of the product of functions having certain degrees is equal to the sum of their degrees.

We use the notations X and T just to emphasize that the functions $S(X,T)$, $n(X,T)$ are "slow" functions of spatial and time variables.

The function $\Phi(\theta, k, \omega, n)$ here represents (for every (k, ω, n)) a one-phase periodic solution of KdV and satisfies the equation

$$\omega\Phi_\theta + k\Phi\Phi_\theta + k^3\Phi_{\theta\theta\theta} = 0 \qquad (1.3)$$

and we put here $S_T = \omega$, $S_X = k$.

We also fix the initial phase shift of the functions $\Phi(\theta, k, \omega, n)$ such that every $\Phi(\theta, k, \omega, n)$ has a local maximum at the point $\theta = 0$ (see Fig. 1).

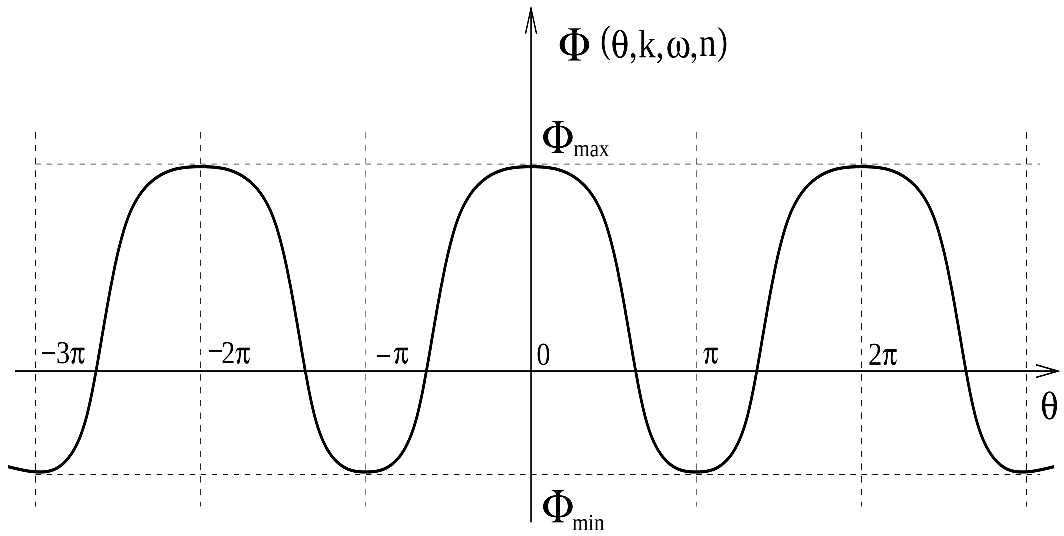

Figure 1: The function $\Phi(\theta, k, \omega, n)$ having zero initial phase shift.

The total function $\Phi^{(tot)}(\theta, X, T)$:

$$\Phi^{(tot)}(\theta, X, T) \;=\; \sum_{l\geq 0} \Phi_{(l)}(\theta, X, T) \;=\; \phi(\theta - S(X,T), X, T)$$

should satisfy the equation

$$\begin{aligned} S_T\Phi^{(tot)}_{\theta} + S_X\Phi^{(tot)}\Phi^{(tot)}_{\theta} + (S_X)^3\,\Phi^{(tot)}_{\theta\theta\theta} + \\ +\Phi^{(tot)}_{T} + \Phi^{(tot)}\Phi^{(tot)}_{X} + 3S_X^2\Phi^{(tot)}_{\theta\theta X} + 3S_XS_{XX}\Phi^{(tot)}_{\theta\theta} + \\ +3S_X\Phi^{(tot)}_{\theta XX} + 3S_{XX}\Phi^{(tot)}_{\theta X} + S_{XXX}\Phi^{(tot)}_{\theta} + \\ +\Phi^{(tot)}_{XXX} \;=\; 0 \end{aligned} \tag{1.4}$$

Equation (1.3) gives the standard representation of one-phase solutions of KdV such that

$$\sqrt{\frac{k^3}{2}}\int_{a_3}\frac{d\Phi}{\sqrt{-k\Phi^3/6 - \omega\Phi^2/2 + A\Phi + B}} \;=\; \theta$$

where a_3 is the third zero of the expression $-k\Phi^3/6 - \omega\Phi^2/2 + A\Phi + B$ according to normalization shown at Fig. 1.

The parameters k, ω, A, B are connected by the equation

$$\sqrt{\frac{k^3}{2}} \oint \frac{d\Phi}{\sqrt{-k\Phi^3/6 - \omega\Phi^2/2 + A\Phi + B}} \;=\; 2\pi$$

which defines finally the function $\Phi(\theta, k, \omega, n)$ depending on 3 parameters where

$$n \;=\; \int_0^{2\pi} \Phi(\theta) \, \frac{d\theta}{2\pi}$$

The function $\Phi_{(1)}(\theta, X, T)$ satisfies the equation

$$\omega\Phi_{(1)\theta} \,+\, k\Phi_{(1)}\Phi_\theta \,+\, k\Phi_{(1)\theta}\Phi \,+\, k^3\Phi_{(1)\theta\theta\theta} \;=\; f_{(1)}(\theta, X, T) \qquad (1.5)$$

where

$$f_{(1)}(\theta, X, T) \;=\; -\,\Phi^{[1]}_T \,-\, \Phi\Phi_X \,-\, 3k^2\Phi_{\theta\theta X} \,-\, 3kk_X\Phi_{\theta\theta} \qquad (1.6)$$

Let us denote by $\hat{L}$ the linear operator

$$\hat{L} \;=\; \omega\,\frac{\partial}{\partial\theta} \,+\, k\,\frac{\partial}{\partial\theta}\,\Phi \,+\, k^3\,\frac{\partial^3}{\partial\theta^3} \qquad (1.7)$$

We can write then (1.6) in the form

$$\hat{L}\,\Phi_{(1)} \;=\; f_{(1)}$$

In the same way we have the analogous systems for the functions $\Phi_{(l)}(\theta, X, T)$ having the form

$$\hat{L}\,\Phi_{(l)} \;=\; \omega\Phi_{(l)\theta} + k\Phi_{(l)}\Phi_\theta + k\Phi_{(l)\theta}\Phi + k^3\Phi_{(l)\theta\theta\theta} \;=\; f_{(l)}(\theta, X, T) \quad (1.8)$$

The functions $k(X,T) = S_X$, $\omega(X,T) = S_T$ and $n(X,T)$ should satisfy the "deformed Whitham system"

$$
\begin{aligned}
k_T &= \omega_X \\
\omega_T &= \sum_{l\geq 1} \sigma_{(l)}\left(k, \omega, n, k_X, \omega_X, n_X, \dots\right) \\
n_T &= \sum_{l\geq 1} \eta_{(l)}\left(k, \omega, n, k_X, \omega_X, n_X, \dots\right)
\end{aligned} \tag{1.9}
$$

where all $\sigma_{(l)}$, $\eta_{(l)}$ are local expressions in $(k, \omega, n, k_X, \omega_X, n_X, \dots)$ polynomial in derivatives and having gradation l.

Easy to see that relations (1.9) give in fact a possibility to represent in the gradated form any expression like $k_{TX\dots X}$, $\omega_{TX\dots X}$, $n_{TX\dots X}$, and even $k_{T\dots TX\dots X}$, $\omega_{T\dots TX\dots X}$, $n_{T\dots TX\dots X}$ using the subsequent substitution of the series (1.9) several times. (The last property will be not necessary in fact for KdV equation).

According to (1.9) all the time derivatives like Φ_T, $\Phi_{(l)T}$ can be also represented as the gradated expansions

$$
\Phi_{(l)T} = \Phi_{(l)T}^{[l]} + \Phi_{(l)T}^{[l+1]} + \Phi_{(l)T}^{[l+2]} + \dots
$$

where the functions $\Phi_{(l)T}^{[s]}$ are also local functionals of $(k, \omega, n, k_X, \omega_X, n_X, \dots)$ polynomial in X-derivatives $(k_X, \omega_X, n_X, \dots)$ and having degree s.

All $f_{(l)}(\theta, X, T)$ should satisfy the orthogonality conditions

$$
\int_0^{2\pi} f_{(l)} \frac{d\theta}{2\pi} = 0 \quad , \quad \int_0^{2\pi} \Phi \; f_{(l)} \frac{d\theta}{2\pi} = 0 \tag{1.10}
$$

We impose also the following "normalization" conditions

$$
\int_0^{2\pi} \Phi_\theta \; \Phi_{(l)} \frac{d\theta}{2\pi} = 0 \quad , \quad \int_0^{2\pi} \Phi_{(l)} \frac{d\theta}{2\pi} = 0 \tag{1.11}
$$

for the functions $\Phi_{(l)}(\theta, X, T)$ defined from(1.8) modulo the linear combinations $a(X,T)\Phi_\theta + b(X,T)\Phi_n$.

For the determination of $\sigma_{(l)}$, $\eta_{(l)}$ we should use system (1.9) to remove all the time derivatives of (k, ω, n) after the substitution of (1.2) into (1.1) such that we can represent (1.4) in the gradated form.

The functions $\sigma_{(l)}$, $\eta_{(l)}$ arising in (1.9) are defined then from the compatibility conditions of systems (1.8) in the l-th order. It can be shown that conditions (1.10)-(1.11) define uniqely all the functions $\sigma_{(l)}$, $\eta_{(l)}$ and the corrections $\Phi_{(l)}$, $l \geq 1$.

Thus we can say that system (1.9) can be defined in the following way:

System (1.9) is the system describing the evolution of parameters (k, ω, n) of zero approximation of (1.2) provided that the following conditions are satisfied:

I) All the functions $\Phi(\theta, k, \omega, n)$ are chosen in the way shown at Fig. 1 ;

II) The modulated phase $S(X, T)$ is connected with the parameters (k, ω, n) by the relations

$$S_T(X, T) = \omega(X, T) \ , \ S_X(X, T) = k(X, T)$$

III) All the higher corrections $\Phi_{(l)}(\theta, X, T)$, $l \geq 1$ satisfy normalization conditions (1.11).

According to our statements above we can claim then that system (1.9) is uniquely defined by conditions (I)-(III).

Lemma 1.

For the "unified" choice of the functions $\Phi(\theta, k, \omega, n)$ corresponding to Fig. 1 the following statements are true:

1) All the even terms $\sigma_{(2l)}(k, \omega, n, \ldots)$, $\eta_{(2l)}(k, \omega, n, \ldots)$ in the deformation of Whitham system (1.9) are identically zero: $\sigma_{(2l)} \equiv 0$, $\eta_{(2l)} \equiv 0$;

2) All the odd corrections $\Phi_{(2l+1)}(\theta, X, T)$, $l \geq 0$ in (1.2) are anti-symmetric in θ: $\Phi_{(2l+1)}(-\theta) = -\Phi_{(2l+1)}(\theta)$;

3) All the even corrections $\Phi_{(2l)}(\theta, X, T)$, $l \geq 1$ in (1.2) are symmetric in θ: $\Phi_{(2l)}(-\theta) = \Phi_{(2l)}(\theta)$.

2 Deformation scheem for the case of small amplitude of oscillations.

Let us say however that the procedure of deformation described above has in fact one weak point. Namely, in the procedure described the higher corrections $\Phi_{(l)}(\theta, X, T)$ as well as the higher deformation terms in system (1.9) are in fact singular in the limit of the small amplitude of oscillations of $\varphi(x, t)$. The reason for such a behavior can be explained in the following way:

Let us rewrite system (1.8) in form

$$\omega\Phi_{(l)} + k\Phi_{(l)}\Phi + k^3\Phi_{(l)\theta\theta} = g_{(l)}(\theta, X, T)$$

where the right-hand part $g_{(l)}$ is periodic in θ view conditions (1.10)

$$g_{(l)}(\theta, X, T) \equiv \int_0^\theta f_{(l)}(\theta', X, T)\, d\theta' + \delta_{(l)}(X, T)$$

We can write this system in the form

$$\hat{Q}_{[X,T]}\Phi_{(l)} = g_{(l)} \tag{2.1}$$

where

$$\hat{Q}_{[X,T]} \equiv \omega(k, A, n) + k\,\Phi + k^3 \frac{\partial^2}{\partial\theta^2} \tag{2.2}$$

is self-adjoint operator on the space of 2π-periodic functions.

Provided that conditions (1.10) are satisfied we can write the solution of (2.1) in the form

$$\Phi_{(l)} \; = \; \sum_j \frac{1}{\lambda_j} \, \xi_j(\theta, X, T) \, \langle \xi_j, g_{(l)} \rangle \qquad (2.3)$$

where $\xi_j(\theta, X, T)$ are normalized eigen-vectors of $\hat{Q}_{[X,T]}$ corresponding to non-zero eigen-values λ_j.

Let us consider now operator (2.2) for the case of small oscillations amplitude:

$$\Phi(\theta, X, T) \; = \; n(X, T) \; + \; a_0(X, T) \, \cos\theta \; + \; \dots \qquad , \quad a_0 \rightarrow 0$$

The parameter $a_0(X, T)$ is the amplitude of the first Fourie harmonic of $\Phi(\theta, X, T)$ which is similar to the parameter $A = \Phi_{max} - \Phi_{min}$ in the limit $A \rightarrow 0$.

Operator (2.2) has always the eigen-vector $\xi(\theta, X, T) \; = \; \Phi_\theta(\theta, X, T)$ corresponding to zero eigen-value, which corresponds to the function $-\sin\theta$ in the limit $A \rightarrow 0$. The dispersion relation $\omega = \omega(k, A, n)$ becomes the dispersion relation of the linear system

$$\omega \; = \; - \, n \, k \; + \; k^3$$

for $A = 0$.

However, the function $\cos\theta$ gives also an eigen-vector of linear operator $(A = 0)$ (2.2) corresponding to zero eigen-value. As a result there exists the eigen-vector $\xi_1(\theta, X, T)$ of operator $\hat{Q}_{[X,T]}$ corresponding to "small" eigen-value $\lambda_1 \rightarrow 0$ (for $A \rightarrow 0$).

Just by direct substitution it's not difficult also to get the following relations for the values of $\omega(k, A, n)$, Φ, $\hat{Q}$, ξ_1 and λ_1:

$$\omega \; = \; - \, k \, n \; + \; k^3 \; - \; \frac{a_0^2}{24k} \; + \; \mathcal{O}(a_0^4)$$

$$\Phi(\theta, k, A, n) \; = \; n \; + \; a_0 \, \cos\theta \; + \; \frac{a_0^2}{12k^2} \cos 2\theta \; + \; \mathcal{O}(a_0^3)$$

$$\hat{Q}_{[k,A,n]} \; = \; k^3 \; - \; \frac{a_0^2}{24k} \; + \; k\, a_0 \, \cos\theta \; + \; \frac{a_0^2}{12k} \cos 2\theta \; + \; k^3 \frac{\partial^2}{\partial\theta^2} \; + \; \mathcal{O}(a_0^3)$$

$$\xi_1(\theta, k, A, n) \; = \; \cos\theta \; - \; \frac{a_0}{2k^2} \; + \; \frac{a_0}{6k^2} \, \cos 2\theta \; + \; \mathcal{O}(a_0^2)$$

$$\lambda_1 \; = \; -\frac{5a_0^2}{12k} \; + \; \mathcal{O}(a_0^4)$$

We can see then that solutions (2.3) become singular in the limit of small oscillations amplitude $A \to 0$ if we don't put additional requirement

$$\langle \xi_1, g_{(l)} \rangle \; \equiv \; 0$$

for all $g_{(l)}$. To improve the deformation procedure described above we should change the deformation scheem for the case of almost linear systems.

Namely, the orthogonality of $g_{(l)}(\theta, X, T)$ to $\xi_1(\theta, X, T)$ can be provided in the following way:

First of all we choose the parameters (k, A, n) instead of (k, ω, n) as the regular parameters everywhere (including the region $A \to 0$). Now the main approximation in the asymptotic solution (1.2) will be given again by the function $\Phi(S(X,T) + \theta, \; k, A, n)$ such that $S_X(X,T) = k(X,T)$. So we have again the same approximation with the same relation between S and k as previosely at every T. However, we make now also the "deformation" of time evolution of phase $S(X,T)$ such that $S_T(X,T) \neq \omega(k, A, n)$ anymore. Instead, we put now the deformed relation

$$S_T \; = \; \omega(k, A, n) \; + \; \sum_{l \geq 1} \omega_{(l)}(k, A, n, k_X, A_X, n_X, \ldots) \qquad (2.4)$$

connecting the time derivative S_T and the parameters (k, A, n) of the main approximation. Here again all the functions $\omega_{(l)}(k, A, n, k_X, A_X, n_X, \ldots)$ are the smooth functions polinomial in derivatives $(k_X, A_X, n_X, \ldots)$ and having degree l according to the same gradation rule, i.e.

All the functions $f(k, A, n)$ have degree 0;

The derivatives k_{lX}, A_{lX}, n_{lX} have degree l;

The degree of the product of functions having certain degrees is equal to the sum of their degrees.

As we said already the parameters $A = \Phi_{max} - \Phi_{min}$ play here the role of the amplitude of oscillations and we have $A(X,T) \sim a_0(X,T)$ for the small A.

We write now the deformed Whitham system in the form

$$\begin{aligned} k_T &= \left(\omega(k,A,n) + \sum_{l\geq 1} \omega_{(l)}(k,A,n,k_X,A_X,n_X,\ldots)\right)_X \\ A_T &= \sum_{l\geq 1} \alpha_{(l)}(k,A,n,k_X,A_X,n_X,\ldots) \\ n_T &= \sum_{l\geq 1} \eta_{(l)}(k,A,n,k_X,A_X,n_X,\ldots) \end{aligned} \tag{2.5}$$

which gives a full deformation of the Whitham system having a regular behavior in the case of small amplitudes.

The functions $\alpha_{(l)}$, $\eta_{(l)}$ are defined as previousely from the orthogonality conditions of the functions $f_{(l)}(\theta,X,T)$ to the "left" eigen-vectors $\Phi(\theta)$ and 1 of the operator $\hat{L}$ corresponding to the zero eigen-values. The functions $\omega_{(l)}$ in (2.4) are defined now from the orthogonality of the functions $g_{(l)}(\theta,X,T)$ to the "left" eigen-vector $\xi_1(\theta,X,T)$ of the operator $\hat{Q}_{[X,T]}$ corresponding to the "small" eigen-value $\lambda_1(k,A,n)$.

So we have the condition

$$\int_0^{2\pi} \xi_1(\theta,X,T)\, g_{(l)}(\theta,X,T)\, \frac{d\theta}{2\pi} \;\equiv\; 0 \tag{2.6}$$

in addition to conditions (1.10) now. The functions $\lambda_1(k,A,n)$, $\xi_1(\theta,k,A,n)$ are defined by continuity on the whole family of one-phase solutions so we can define the system (2.5) on the full space of parameters.

It is easy to prove for our choice of the functions $\Phi(\theta,k,A,n)$ that the function $\xi_1(\theta,k,A,n)$ is symmetric in θ, i.e. $\xi_1(-\theta,k,A,n) = \xi_1(\theta,k,A,n)$.

For the solutions $\Phi_{(l)}(\theta, X, T)$ we will have automatically

$$\int_0^{2\pi} \xi_1(\theta, X, T)\, \Phi_{(l)}(\theta, X, T)\, \frac{d\theta}{2\pi} \;\equiv\; 0 \tag{2.7}$$

in addition to normalization conditions (1.11).

In the same way as previousely the following lemma can be proved for systems (2.4)-(2.5) and the asymptotic expansion

$$\phi(\theta, X, T) \;=\; \Phi(S(X,T) + \theta, k, A, n) \;+\; \sum_{l \geq 1} \Phi_{(l)}(S(X,T) + \theta, X, T) \tag{2.8}$$

Lemma 2.

For the "unified" choice of the functions $\Phi(\theta, k, A, n)$ corresponding to Fig. 1 the following statements are true:

1) All the even terms $\sigma_{(2l)}(k, A, n, \ldots)$, $\eta_{(2l)}(k, A, n, \ldots)$ in the deformation of Whitham system (2.5) are identically zero: $\alpha_{(2l)} \equiv 0$, $\eta_{(2l)} \equiv 0$;

2) All the odd terms $\omega_{(2l+1)}(k, A, n, \ldots)$, $l \geq 0$ in the deformation (2.4) of dispersion relation are identically zero: $\omega_{(2l+1)} \equiv 0$;

3) All the odd corrections $\Phi_{(2l+1)}(\theta, X, T)$, $l \geq 0$ in (1.2) are anti-symmetric in θ: $\Phi_{(2l+1)}(-\theta) = -\Phi_{(2l+1)}(\theta)$;

4) All the even corrections $\Phi_{(2l)}(\theta, X, T)$, $l \geq 1$ in (1.2) are symmetric in θ: $\Phi_{(2l)}(-\theta) = \Phi_{(2l)}(\theta)$.

So we can rewrite relation (2.4) and system (2.5) in the form

$$S_T \;=\; \omega(k, A, n) \;+\; \sum_{l \geq 1} \omega_{(2l)}(k, A, n, k_X, A_X, n_X, \ldots) \tag{2.9}$$

$$k_T \;=\; \left(\omega(k, A, n) \;+\; \sum_{l \geq 1} \omega_{(2l)}(k, A, n, k_X, A_X, n_X, \ldots)\right)_X$$

$$A_T = \sum_{l \geq 0} \alpha_{(2l+1)}(k, A, n, k_X, A_X, n_X, \dots) \qquad (2.10)$$

$$n_T = \sum_{l \geq 0} \eta_{(2l+1)}(k, A, n, k_X, A_X, n_X, \dots)$$

We can see that for our choice of the functions $\Phi(\theta, k, A, n)$ the full deformation (2.10) of the Whitham system includes only odd degrees of the expansion in higher derivatives which emphasizes the dispersive character of the deformation.

System (2.10) can be defined as the system giving the evolution of parameters (k, A, n) of zero approximation of (1.2) such that the following conditions are satisfied:

I′) All the functions $\Phi(\theta, k, A, n)$ are chosen in the way shown at Fig. 1 ;

II′) The modulated phase $S(X, T)$ and the parameters (k, A, n) are connected by the relation

$$S_X(X, T) = k(X, T)$$

III′) All the higher corrections $\Phi_{(l)}(\theta, X, T)$, $l \geq 1$ satisfy normalization conditions (1.11) and (2.7).

We would like to introduce a small parameter ϵ according to our gradation rule for more convienent notations. System (2.9)-(2.10) will be rewritten then in the form

$$S_T = \omega(k, A, n) + \sum_{l \geq 1} \epsilon^{2l} \, \omega_{(2l)}(k, A, n, k_X, A_X, n_X, \dots) \qquad (2.11)$$

$$k_T = \left(\omega(k, A, n) + \sum_{l \geq 1} \epsilon^{2l} \, \omega_{(2l)}(k, A, n, k_X, A_X, n_X, \dots) \right)_X$$

$$A_T \;=\; \sum_{l\geq 0} \epsilon^{2l}\, \alpha_{(2l+1)}(k, A, n, k_X, A_X, n_X, \dots) \qquad (2.12)$$

$$n_T \;=\; \sum_{l\geq 0} \epsilon^{2l}\, \eta_{(2l+1)}(k, A, n, k_X, A_X, n_X, \dots)$$

Asymptotic expansion (2.8) should be also rewritten in the form

$$\phi(\theta, X, T) \;=\; \Phi\left(\frac{S(X,T)}{\epsilon} + \theta, k, A, n\right) + \qquad (2.13)$$

$$+ \sum_{l\geq 1} \epsilon^{l}\, \Phi_{(l)}\left(\frac{S(X,T)}{\epsilon} + \theta, X, T\right)$$

according to the gradation rule of the functions $S(X,T)$ and $\Phi_{(l)}(\theta, X, T)$.

In these new notations we can see actually that system (2.15)-(2.16) describes the asymptotic solutions of the equation

$$\phi_T \,+\, \phi\,\phi_X \,+\, \epsilon^2\, \phi_{XXX} \;=\; 0 \qquad (2.14)$$

where the small dispersion ϵ^2 arises after the stretch of coordinates $T \longrightarrow \epsilon T$, $X \longrightarrow \epsilon X$. (Easy to see that the ϵ-rescaling of θ does not change equation (2.14)). So our futher considerations will be apllied to the KdV equation in the small-dispersion form (2.14).

Let us emphasize, however, that (2.13) is not an ϵ-expansion of the asymptotic solution of (2.14) since all of the functions $k(X,T,\epsilon)$, $A(X,T,\epsilon)$, $n(X,T,\epsilon)$ are solutions of ϵ-dependent system (2.12), such that expansion (2.13) can contain more complicated ϵ-dependence. According to our rules we should not divide the different orders of ϵ in the functions $k(X,T,\epsilon)$, $A(X,T,\epsilon)$, $n(X,T,\epsilon)$ and just use the gradation rules formulated above for the ϵ-dependent functions.

Let us say finally that we are going to consider the system

$$S_T \;=\; \omega(k, A, n) \,+\, \epsilon^2\, \omega_{(2)}(k, A, n, \dots) \qquad (2.15)$$

$$
\begin{aligned}
k_T &= \left(\omega(k,A,n) + \epsilon^2\,\omega_{(2)}(k,A,n,\ldots)\right)_X \\
A_T &= a_{(1)}(k,A,n,k_X,A_X,n_X) + \epsilon^2\,a_{(3)}(k,A,n,\ldots) \\
n_T &= \eta_{(1)}(k,A,n,k_X,A_X,n_X) + \epsilon^2\,\eta_{(3)}(k,A,n,\ldots)
\end{aligned}
\tag{2.16}
$$

since we believe that it demonstrates already many of the sufficient features of full system (2.11)-(2.12).

An important procedure is the procedure of "averaging" of local Hamiltonian structures suggested by B.A. Dubrovin and S.P. Novikov. The Dubrovin - Novikov procedure gives a field-theoretical Hamiltonian structure of Hydrodynamic Type for the Whitham system with a Hamiltonian function having the hydrodynamic form $H = \int h(\mathbf{U})dX$. The Dubrovin - Novikov bracket has the form

$$
\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(\mathbf{U})\,\delta'(X-Y) + b^{\nu\mu}_\lambda(\mathbf{U})\,U^\lambda_X\,\delta(X-Y) \tag{2.17}
$$

which is called also a local Poisson bracket of Hydrodynamic Type.

The Hamiltonian properties of systems of Hydrodynamic Type are strongly correlated with their integrability properties. Thus it was proved by S.P. Tsarev that all the diagonalizable systems of Hydrodynamic Type having the Dubrovin - Novikov Hamiltonian structure can in fact be integrated (S.P. Novikov conjecture).

Deformation of the Whitham system implies also the deformation of the corresponding Hamiltonian structures (2.17)

$$
\begin{aligned}
\{U^\nu(X), U^\mu(Y)\} &= \{U^\nu(X), U^\mu(Y)\}_0 + \\
+ \sum_{k\geq 2} \epsilon^{k-1} \sum_{s=0}^{k} & B^{\nu\mu}_{(k)s}(\mathbf{U}, \mathbf{U}_X, \ldots, \mathbf{U}_{(k-s)X})\,\delta^{(s)}(X-Y)
\end{aligned}
\tag{2.18}
$$

where all $B^{\nu\mu}_{(k)s}$ are polynomial w.r.t. derivatives $\mathbf{U}_X$, ..., $\mathbf{U}_{(k-s)X}$ and have degree $(k-s)$.

We call deformations of Hamiltonian structure (2.17) of form (2.18) the deformations of Dubrovin-Zhang type.

Let us formulate here the following Lemma:

Lemma 3.

The Gardner-Zakharov-Faddeev bracket and the Magri bracket for the KdV equation give the brackets of the form (2.18) for the deformed Whitham system after the "extended" procedure of averaging of a Poisson bracket.