AUTOPHASING OF SOLITONS

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The problem

$$\underline{NS\ equation}: \quad iu_t + \frac{1}{2}u_{xx} + |u|^2 u = \varepsilon e^{ikx + i\psi(t)}$$

$$\underline{SG\ equation}: \quad u_{tt} - u_{xx} + \sin u = \varepsilon \cos\psi(t)$$

$$0 < \varepsilon \ll 1, \quad \psi_t = \Omega(t), \quad \Omega_t = O(\alpha), \quad |\alpha| \ll 1$$

$$\underline{NS\ soliton}: \qquad u = \varphi_s(x,t) = \frac{A}{\cosh\left[A(x-\xi)\right]} e^{i[V(x-\xi)+\theta]},$$

$$\varepsilon = 0: \quad \xi = Vt + \xi_0, \quad \theta = \omega t + \theta_0, \quad \omega = \frac{1}{2}(A^2 + V^2)$$

$$\underline{SG\ breather}: \quad u = \varphi_b(x,t) = -4\arctan\left[\tan A \frac{\cos\theta}{\cosh\left(x \sin A\right)}\right],$$

$$\varepsilon = 0: \quad \theta = \omega t + \theta_0, \quad \omega = \cos A$$

In the presence of pumping we set

$$u(x,t) = \varphi(x,t) + \chi(x,t), \qquad \varphi(x,t) \to 0 \quad (|x| \to \infty)$$

and obtain the perturbed NS and SG equations:

$$i\varphi_t + \frac{1}{2}\varphi_{xx} + |\varphi|^2 \varphi = -\chi^* \varphi^2 - 2\chi |\varphi|^2$$

$$i\chi_t + \frac{1}{2}\chi_{xx} = \varepsilon e^{ikx + i\psi(t)}$$

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = \chi(t)(1 - \cos\varphi)$$

$$\chi_{tt} + \chi = \varepsilon \cos\psi(t)$$

In the adiabatic approximation

$$\varphi \approx \varphi_s(x,t;A,V,\xi,\theta), \quad \varphi \approx \varphi_b(x,t;A,\theta)$$

Equations for the NS soliton parameters

$$
\begin{aligned}
A_t &= -\varepsilon \frac{A^2}{2\Omega + k^2} F\left(\frac{V-k}{A}\right) \sin\delta, \\
V_t &= \varepsilon \frac{A(V-k)}{2\Omega + k^2} F\left(\frac{V-k}{A}\right) \sin\delta, \\
\xi_t &= V + \frac{\varepsilon}{2\Omega + k^2} F'\left(\frac{V-k}{A}\right) \cos\delta, \\
\delta_t &= \Delta\omega - \varepsilon \frac{2A}{2\Omega + k^2} \left[F\left(\frac{V-k}{A}\right) - \frac{V-k}{A} F'\left(\frac{V-k}{A}\right) \right] \cos\delta,
\end{aligned}
$$

where

$$\delta = \theta - \psi - k\xi, \quad \Delta\omega = \omega - \Omega - kV, \quad \omega = (A^2 + V^2)/2, \quad F(x) = \pi(1 + x^2)\,\mathrm{sech}\,(\pi x/2)$$

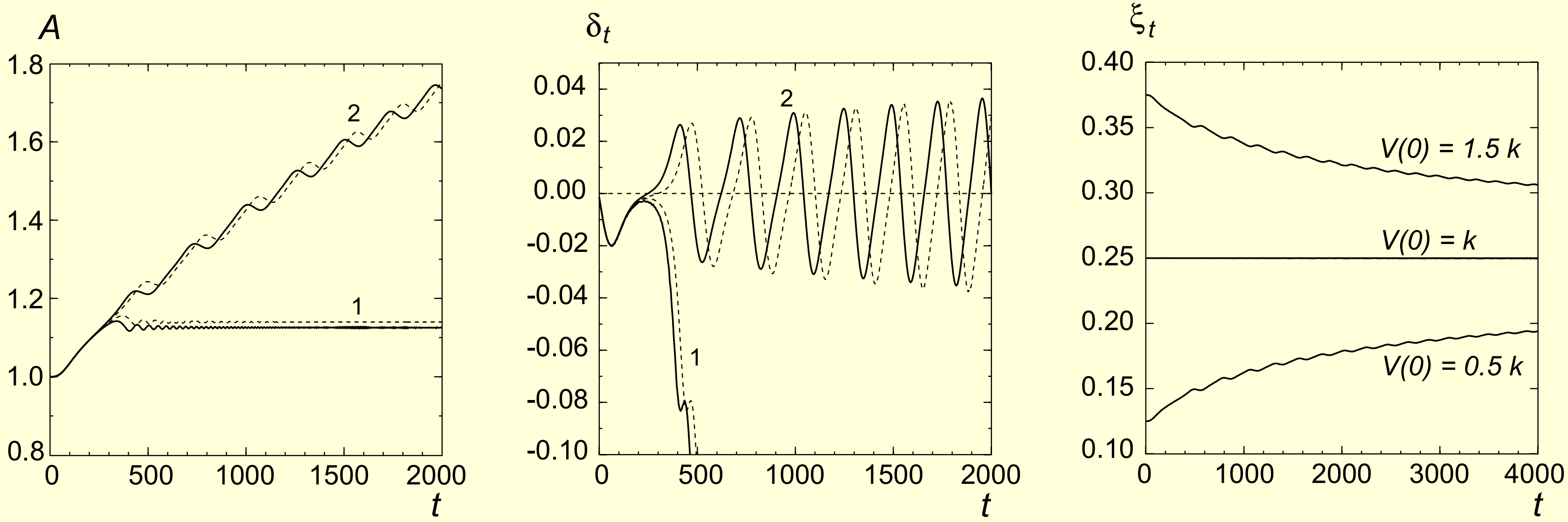
Conservation of the "momentum":

$$P = A(V-k) = const$$

The system reduces to the Hamiltonian equations:

$$\begin{aligned} A_t &= -\varepsilon \frac{f(A)}{2\Omega + k^2} \sin\delta, \\ \delta_t &= \Delta\omega - \varepsilon \frac{f'(A)}{2\Omega + k^2} \cos\delta, \end{aligned}$$

where $f(A) = A^2 F(P/A^2), \quad \Delta\omega = (A^2 - k^2 + P^2/A^2)/2 - \Omega$



For $\Omega(t) = \Omega_0 + \alpha t$ with $\alpha = 5 \times 10^{-4}$ we obtain
$\varepsilon_{cr}^{theor} = 2.18198 \times 10^{-4}$, while $\varepsilon_{cr}^{NS} = 2.1844 \times 10^{-4}$

Equations for the SG breather parameters

$$A_t = \varepsilon \frac{Q(A, \psi + \delta)}{1 - \Omega^2} \cos \psi,$$

$$\delta_t = \Delta\omega(A, \Omega) + \varepsilon \frac{R(A, \psi + \delta)}{1 - \Omega^2} \cos \psi,$$

where $\delta = \theta - \psi, \quad \psi_t = \Omega, \quad \Delta\omega = \omega - \Omega, \quad \omega = \cos A, \quad \varkappa = \sin A,$

$$Q(A, \theta) = \frac{\pi}{4} \varkappa^2 \sin\theta \cos^2\theta \frac{4\omega^2 + \varkappa^2 \cos^2\theta}{(1 - \varkappa^2 \sin^2\theta)^{5/2}},$$

$$R(A, \theta) = \frac{\pi\omega}{4\varkappa^2} \left\{ \frac{\omega^2 \varkappa \cos\theta}{(1 - \varkappa^2 \sin^2\theta)^{3/2}} \left[1 + \frac{\varkappa^2 \cos^2\theta (4\omega^2 + \varkappa^2 \cos^2\theta)}{\omega^4 (1 - \varkappa^2 \sin^2\theta)} \right] - \operatorname{arsh}\left(\frac{\varkappa}{\omega} \cos\theta \right) \right\}$$

Generalized Krylov-Bogoliubov averaging procedure: $(A,\delta) \rightarrow (\bar{A},\bar{\delta})$

$$\begin{pmatrix} A \\ \delta \end{pmatrix} = \begin{pmatrix} \bar{A} \\ \bar{\delta} \end{pmatrix} + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \left(\overline{\Delta\omega}\right)^n \begin{pmatrix} u_{mn}\left(\bar{A},\bar{\delta},\Omega,\psi\right) \\ v_{mn}\left(\bar{A},\bar{\delta},\Omega,\psi\right) \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} \bar{A} \\ \bar{\delta} \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\Delta\omega} \end{pmatrix} + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \left(\overline{\Delta\omega}\right)^n \begin{pmatrix} A_{mn}\left(\bar{A},\bar{\delta},\Omega\right) \\ B_{mn}\left(\bar{A},\bar{\delta},\Omega\right) \end{pmatrix}$$

The first-order averaged system:

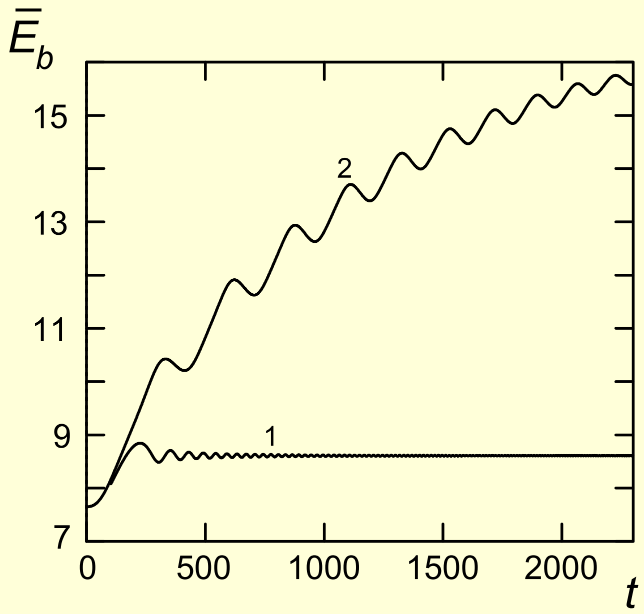
$$\boxed{\begin{aligned} \bar{A}_t &= \varepsilon\frac{g(\bar{A})}{1-\Omega^2(t)}\sin\bar{\delta}, \\ \bar{\delta}_t &= \overline{\Delta\omega} + \varepsilon\frac{g'(\bar{A})}{1-\Omega^2(t)}\cos\bar{\delta}, \end{aligned}}$$

where

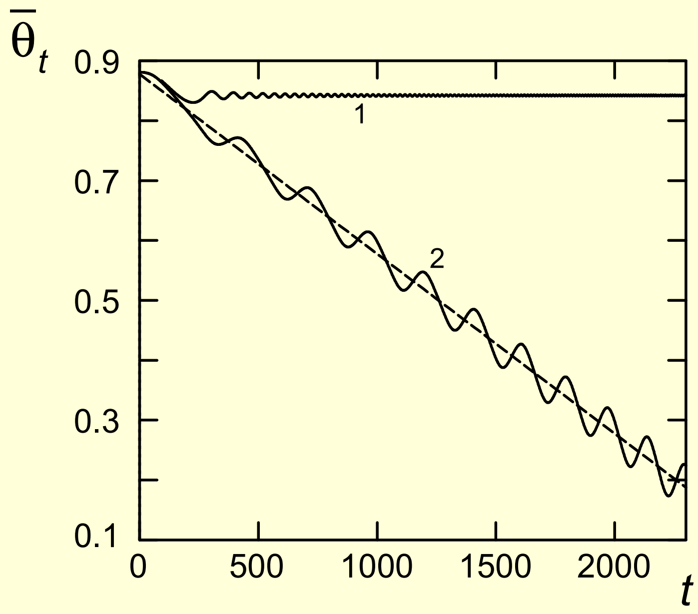
$$\overline{\Delta\omega} = \cos\bar{A} - \Omega, \quad g(\bar{A}) = \frac{1}{2}\left[K(\sin\bar{A}) - E(\sin\bar{A})\right] \geqslant 0, \quad g'(\bar{A}) = \frac{1}{2}E(\sin\bar{A})\,\mathrm{tg}\,\bar{A} \geqslant 0$$

Numerical integration gives

Energy: $\bar{E}_b = 16 \sin \bar{A}$



Frequency: $\bar{\theta}_t = \Omega(t) + \bar{\delta}_t$



For $\Omega(t) = \Omega_0 + \alpha t$ with $\alpha = -0.0003$ we obtain $\varepsilon_{cr}^{theor} = 1.78 \times 10^{-3}$, while $\varepsilon_{cr}^{SG} = 1.79 \times 10^{-3}$.

General structure of equations describing autophasing

$$\begin{aligned} A_t &= \varepsilon G(A;\Omega)\sin\delta, \\ \delta_t &= \Delta\omega(A;\Omega) + \varepsilon \frac{\partial G(A;\Omega)}{\partial A}\cos\delta \end{aligned}$$

<u>NS</u>:

$$\begin{aligned} G(A;\Omega) &= -\frac{f(A)}{2\Omega + k^2}, \\ \Delta\omega(A;\Omega) &= \frac{1}{2}(A^2 - k^2 + P^2/A^2) - \Omega \end{aligned}$$

<u>SG</u>:

$$A \to \bar{A}, \quad \delta \to \bar{\delta}, \quad \Delta\omega \to \overline{\Delta\omega} = \cos\bar{A} - \Omega$$

$$G(\bar{A};\Omega) = \frac{g(\bar{A})}{1-\Omega^2}$$

Nonlinear pendulum approximation

$$\delta_{tt} = \varepsilon \left(\frac{\partial \Delta\omega}{\partial A} G - \Delta\omega \frac{\partial G}{\partial A} \right) \sin\delta - \Omega_t + O(\varepsilon^2, \alpha\varepsilon)$$

At the initial time interval $0 \leqslant t \ll O(\varepsilon^{-1})$

$$\delta_{\tau\tau} = \lambda \sin\delta - \beta, \qquad (0 \leqslant \tau \ll O(\varepsilon^{-1/2})),$$

where $\tau = \varepsilon^{1/2} t$, $\lambda \simeq const$, and we set $\Omega_t = \alpha$, $\beta = \alpha/\varepsilon$.

If $\Delta\omega(t=0) \lesssim O(\varepsilon)$, then

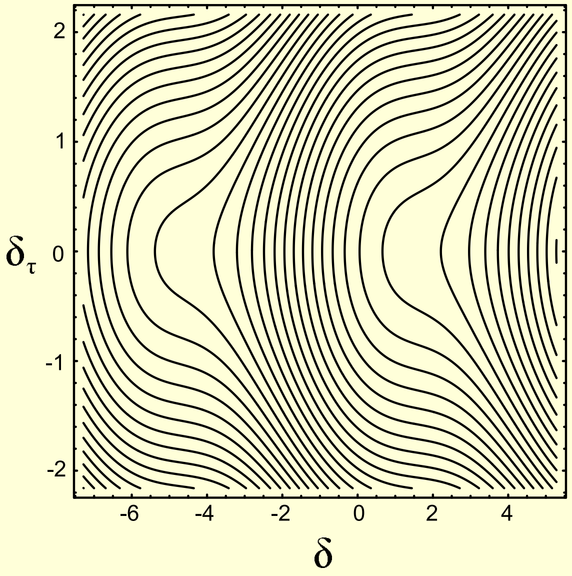
<u>NS</u>:

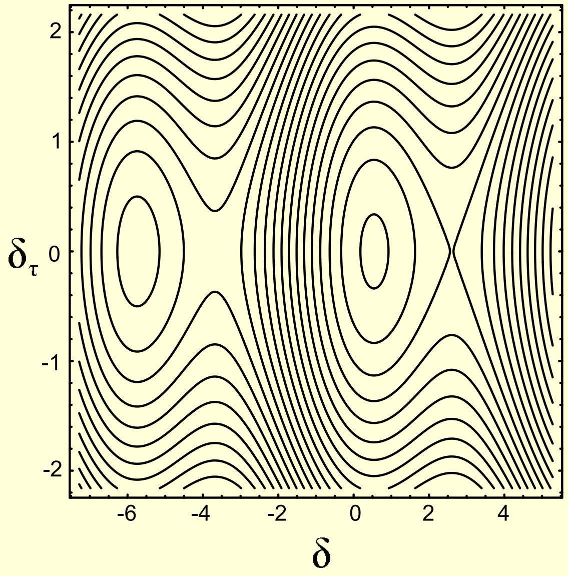
$$\lambda \simeq \left[\frac{f(A)(P^2 - A^4)}{A(P^2 + A^4)} \right]_{t=0}$$

<u>SG</u>:

$$\lambda \simeq -\left[\frac{g(A)}{\sin A} \right]_{t=0} < 0$$

$$E(\delta,\delta_\tau) = \delta_\tau^2/2 + V(\delta), \quad V(\delta) = \lambda\cos\delta + \beta\delta$$





Conditions of phaselocking:

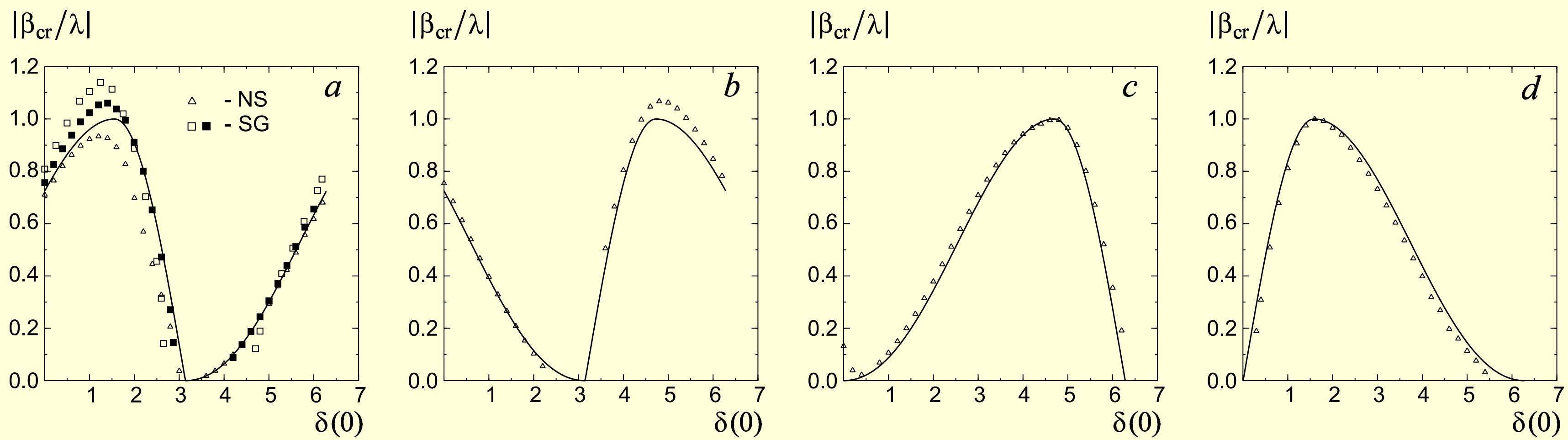
$$\begin{aligned} |\beta/\lambda| &< 1 \\ \delta_n^{(1)} &< \delta(0) < \delta_n^{(2)} \\ E_n^{(0)} &\leqslant E(\delta(0),\delta_\tau(0)) < E_n^{(s)} \end{aligned}$$

Equation for β_{cr}:

$$E\left(\delta(0),\delta_\tau(0);\beta,\lambda\right) = E_n^{(s)}\left(\beta,\lambda\right)$$

$$\cos\delta(0) + \frac{\beta}{\lambda}\delta(0) = -\sqrt{1-\left(\frac{\beta}{\lambda}\right)^2} + \frac{\beta}{\lambda}\left((2n+1)\pi - \arcsin\frac{\beta}{\lambda}\right) \qquad (\lambda < 0)$$

$$\cos\delta(0) + \frac{\beta}{\lambda}\delta(0) = \sqrt{1-\left(\frac{\beta}{\lambda}\right)^2} + \frac{\beta}{\lambda}\left(2n\pi + \arcsin\frac{\beta}{\lambda}\right) \qquad (\lambda > 0)$$



$\lambda < 0, \beta < 0$ $\lambda < 0, \beta > 0$ $\lambda > 0, \beta < 0$ $\lambda > 0, \beta > 0$

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