RIEMANN SPACES ASSOCIATED WITH NONLINEAR EQUATIONS

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The Ricci flat 8-dim metric and KP-equation

In the papers (DR1-DR2) was showed that the eight-dimensional metric of the space in local coordinates (x, y, z, t, P, Q, U, V)

$$ {}^{8}ds^2 = 2\left(-PH_{11} - P\frac{\partial F}{\partial t} - H_{12}Q - 2\Gamma^3_{11}U + H_{22}V\right)dx^2 + $$

$$ +4\left(H_{11}Q - H_{12}V\right)dxdy + 4\left(-\frac{\partial F}{\partial z}V + \frac{\partial F}{\partial t}U\right)dxdz + $$

$$ +2(H_{11}U - H_{31}V)dy^2 + 2dxdP + 2dydQ + 2dzdU + 2dtdV, \quad (1) $$

is the Ricci-flat $R_{ik} = 0$ if the conditions on the functions $H_{ij} = H_{ij}(y, z, t)$, $\Gamma^3_{11}(y, z, t)$ and $F(y, z, t)$

$$ \frac{\partial H_{12}}{\partial y} - \frac{\partial H_{22}}{\partial t} = 0, \quad -\frac{\partial H_{11}}{\partial y} + \frac{\partial H_{21}}{\partial t} = 0, \quad -\frac{\partial H_{11}}{\partial z} + \frac{\partial H_{31}}{\partial t} = 0, \quad (2) $$

$$ \frac{\partial \Gamma^3_{11}}{\partial z} = 2\left(\frac{\partial F}{\partial t}\right)^2 + 2H_{11}\frac{\partial F}{\partial t} + 2(H_{11})^2. \quad (3) $$

are hold.

The metric (1) has the form

$$ds^2 = -\Gamma^i_{jk}\xi_i dx^j dx^k + 2d\xi_k dx^k \qquad (4)$$

and it is an example of Riemann extension of affinely-connected the four dimensional space in local coordinates x^l with symmetric connection $\Gamma^i_{jk}(x^l) = \Gamma^i_{kj}(x^l)$. From the system (2) after the substitution (Konop)

$$H_{11} = -\frac{1}{2}u(y,z,t), \quad H_{12} = -\frac{1}{3}v(y,z,t), \quad H_{21} = -\frac{2}{3}v(y,z,t) - $$

$$-\frac{1}{2}\frac{\partial u(y,z,t)}{\partial t}$$

$$H_{31} = -\frac{3}{4}w(y,z,t) + \frac{3}{8}u(y,z,t)^2 - \frac{\partial v(y,z,t)}{\partial t} - \frac{1}{2}\frac{\partial^2 u(y,z,t)}{\partial t^2},$$

$$H_{22} = -\frac{1}{2}w(y,z,t) + \frac{1}{2}u(y,z,t)^2 - \frac{\partial v(y,z,t)}{\partial t} -$$

$$-\frac{1}{2}\frac{\partial^2 u(y,z,t)}{\partial t^2} + \frac{\partial u(y,z,t)}{\partial y},$$

the famous KP-equation is followed

$$\frac{\partial}{\partial t}\left(\frac{\partial u(y,z,t)}{\partial z}-\frac{3}{2}u(y,z,t)\frac{\partial u(y,z,t)}{\partial t}-\frac{1}{4}\frac{\partial^3 u(y,z,t)}{\partial t^3}\right)=$$

$$=\frac{3}{4}\frac{\partial^2 u(y,z,t)}{\partial y^2}. \tag{5}$$

Four dimensional Ricci-flat affinely connected subspace and KP-equation

Main result of [2] is following

Theorem The four-dimensional affinely connected space with non zero coefficients of connection $\Gamma^i_{jk} = \Gamma^i_{kj}$ of the form

$$\Gamma^1_{11} = H_{11} + \frac{\partial F}{\partial t}, \quad \Gamma^2_{11} = H_{12}, \quad \Gamma^3_{11} = \Gamma^3_{11},, \quad \Gamma^2_{12} = H_{11}, \quad \Gamma^3_{13} = -\frac{\partial F}{\partial t}$$

$$\Gamma^3_{22} = -H_{11}, \quad \Gamma^4_{11} = -H_{22}, \quad \Gamma^4_{12} = H_{21}, \quad \Gamma^4_{13} = \frac{\partial F}{\partial z}, \quad \Gamma^4_{22} = H_{31}$$

is a Ricci flat

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{im}\Gamma^m_{kj} = 0,$$

if the conditions (2-3) hold.
The problem of metrizability such type of connection is important in theory of 3-dim manifolds is an open question.

Six- dimensional Ricci-flat defined by the KP-equation

As particular case we consider the metrics (4) of the form

$$
{}^{6}ds^2 =
$$

$$
= 4\left(\frac{\partial}{\partial x}u(x,y,t)\right)P dx\, dt + 2\, dx\, dP + 4\left(\frac{\partial}{\partial y}u(x,y,t)\right)P dy\, dt + 2\, dy\, dQ +
$$

$$
+\left(-2\,Pu(x,y,t)\frac{\partial}{\partial x}u(x,y,t) - 2\,P\frac{\partial^3}{\partial x^3}u(x,y,t)\right)dt^2 +
$$

$$
+\left(-2\,\mu\left(\frac{\partial}{\partial y}u(x,y,t)\right)Q + 2\left(\frac{\partial}{\partial x}u(x,y,t)\right)U\right)dt^2 +
$$

$$
+2\, dt\, dU. \tag{6}
$$

The Ricci tensor such type of the metric

$$R_{ij} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial(u_t+uu_x+u_{xxx})}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has one component and it is equal to zero on solutions of the KP-equation

$$\frac{\partial(u_t + uu_x + u_{xxx})}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = 0.$$

Remark

The metric (6) arises from the Riemann extension of the 3D Einstein-Weyl space

$$^{3}ds^2 = dy^2 - 4dxdt - 4u(x,y,t)dt^2, \quad \nu = -4u_x dt,$$

associated with the DKP-equation

$$\frac{\partial(u_t - uu_x)}{\partial x} = \frac{\partial^2 u}{\partial y^2},$$

which is obtained from the Einstein-Weyl conditions on the Ricci tensor

$$R_{ij} + 1/2\nabla(_i\nu_j) + 1/4\nu_i\nu_j - 1/3(R + 1/2\nabla^k\nu_k + 1/4\nu^k\nu_k)h_{ij} = 0$$

6D Ricci-flat metrics associated with KdF-equation

3D-Riemann metric of the form

$$ds^2 = y^2 dx^2 + \left(l\,(x,z)\,y^2 - 1/2\right) dx\,dz + 2\,dy\,dz +$$

$$+ \left(\left(l\,(x,z)\right)^2 y^2 - 2\,\left(\frac{\partial}{\partial x} l\,(x,z)\right) y + l\,(x,z)\right) dz^2 \qquad (7)$$

is a flat $R_{ijkl} = 0$ if the function $l(x,z)$ satisfies the KdF-equation (Dryuma,2006)

$$\frac{\partial^3}{\partial x^3} l\,(x,z) + \frac{\partial}{\partial z} l\,(x,z) - 3\,l\,(x,z)\,\frac{\partial}{\partial x} l\,(x,z) = 0. \qquad (8)$$

It has 13 Christoffel symbols.
Some of them are in form

$$\Gamma^1_{11} = 1/2\,\frac{-1+2\,l\,(x,z)\,y^2}{y},\quad \Gamma^1_{12} = y^{-1},\; \Gamma^1_{13} = 1/2\,\frac{\left(-1+2\,l\,(x,z)\,y^2\right)\,l}{y}$$

$$\Gamma^1_{23} = \frac{l\,(x,z)}{y},$$

$$\Gamma^1_{33} = 1/2\,\frac{2\,\left(\frac{\partial}{\partial z}l\,(x,z)\right)\,y - 4\,l\,(x,z)\,y\frac{\partial}{\partial x}l\,(x,z) + 2\,\frac{\partial^2}{\partial x^2}l\,(x,z) - (l\,(x,z))^2}{y}$$

$$\Gamma^2_{11} = -1/4\,\frac{4\,y^3\frac{\partial}{\partial x}l\,(x,z) - 8\,l\,(x,z)\,y^2 + 1}{y}.$$

...

$$\Gamma^3_{11} = -y,\quad \Gamma^3_{13} = -l\,(x,z)\,y,\quad \Gamma^3_{33} = -\,(l\,(x,z))^2\,y + \frac{\partial}{\partial x}l\,(x,z)\,.$$

Six-dimensional Riemann extension of the metrics (7) is the space in local coordinates $(x, y, z, \xi_1, \xi_2, \xi_3$ and it is defined by the expression

$$ds^2 = -2\Gamma^k_{ij}\xi_k dx^i dx^j + 2dx^k\xi_k \tag{9}$$

with a given coefficients Γ^k_{ij}.
Such metric is a flat on solutions of the KdF-equation (8).
To obtain an examples of non a flat $R_{ijkl} \neq 0$ the six-dimensional metrics associated with the KdF-equation it is necessary to introduce an additional terms into the expression (9).
As example the change of the component of metric

$$g_{13} := l(x, z)y^2 - 1/2$$

on the following

$$g_{1,3} := l(x, z)y^2 - 1$$

change radically the Ricci-tensor of the space.
In a six-dimensional case it is possible to change components of metric such a way that metric will be Ricci flat $R_{ik} = 0$ on solutions of the KdF-equation but not a flat $R_{ijkl} \neq 0$.

Six-dim Heavenly metric and Special Lagrangian equation

We study six-dimensional generalization of the Heavenly metrics

$$ds^2 = dx\,du + dy\,dv + dz\,dw + A(x,y,z,u,v,w)du^2 +$$

$$+2\,B(x,y,z,u,v,w)du\,dv + 2\,E(x,y,z,u,v,w)du\,dw + C(\vec{x})dv^2 +$$

$$+2\,H(x,y,z,u,v,w)dv\,dw + F(x,y,z,u,v,w)dw^2. \qquad (10)$$

The Ricci tensor of the metric (10) has a fifteen components. Nine of them are equal to zero due the conditions

$$\frac{\partial}{\partial u}E(x,y,z,u,v,w) + \frac{\partial}{\partial v}H(x,y,z,u,v,w) + \frac{\partial}{\partial w}F(x,y,z,u,v,w) = 0,$$

$$\frac{\partial}{\partial u}B(x,y,z,u,v,w) + \frac{\partial}{\partial v}C(x,y,z,u,v,w) + \frac{\partial}{\partial w}H(x,y,z,u,v,w) = 0,$$

$$\frac{\partial}{\partial u}A(x,y,z,u,v,w) + \frac{\partial}{\partial v}B(x,y,z,u,v,w) + \frac{\partial}{\partial w}E(x,y,z,u,v.w) = 0. \qquad (11)$$

This system of equation has a solutions depending from an arbitrary functions.
In a simplest case we have the solution

$$A(\vec{x}) = \left(\frac{\partial^2}{\partial z^2} f(\vec{x})\right) \frac{\partial^2}{\partial y^2} f(\vec{x}) - \left(\frac{\partial^2}{\partial y \partial z} f(\vec{x})\right)^2,$$

$$C(\vec{x}) = \left(\frac{\partial^2}{\partial x^2} f(\vec{x})\right) \frac{\partial^2}{\partial z^2} f(\vec{x}) - \left(\frac{\partial^2}{\partial x \partial z} f(x, y, z, u, v, w)\right)^2,$$

$$F(\vec{x}) = \left(\frac{\partial^2}{\partial x^2} f(\vec{x})\right) \frac{\partial^2}{\partial y^2} f(\vec{x}) - \left(\frac{\partial^2}{\partial x \partial y} f(\vec{x})\right)^2,$$

$$E(\vec{x}) = \left(\frac{\partial^2}{\partial y \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial x \partial y} f(\vec{x}) - \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial y^2} f(\vec{x})$$

$$B(\vec{x}) = \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial y \partial z} f(\vec{x}) - \left(\frac{\partial^2}{\partial x \partial y} f(\vec{x})\right) \frac{\partial^2}{\partial z^2} f(\vec{x})$$

$$H(\vec{x}) = \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial x \partial y} f(\vec{x}) - \left(\frac{\partial^2}{\partial y \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial x^2} f(\vec{x})$$

depending from one arbitrary function.

At this conditions the six-dimensional metric looks as

$$
\begin{aligned}
{}^{6}ds^2 = &\left(\frac{\partial^2}{\partial w^2}K(\vec{x})\frac{\partial^2}{\partial v^2}K(\vec{x}) - \left(\frac{\partial^2}{\partial v \partial w}K(\vec{x})\right)^2\right) dx^2 + \\
&+2\left(\frac{\partial^2}{\partial u \partial w}K(\vec{x})\frac{\partial^2}{\partial v \partial w}K(\vec{x}) - \frac{\partial^2}{\partial w^2}K(\vec{x})\frac{\partial^2}{\partial u \partial v}K(\vec{x})\right) dxdy + \\
&+\left(\frac{\partial^2}{\partial u^2}K(\vec{x})\frac{\partial^2}{\partial w^2}K(\vec{x}) - \left(\frac{\partial^2}{\partial u \partial w}K(\vec{x})\right)^2\right) dy^2 + \\
&+2\left(\frac{\partial^2}{\partial v \partial w}K(\vec{x})\frac{\partial^2}{\partial u \partial v}K(\vec{x}) - \frac{\partial^2}{\partial u \partial w}K(\vec{x})\frac{\partial^2}{\partial v^2}K(\vec{x})\right) dxdz + \\
&+\left(\frac{\partial^2}{\partial v^2}K(\vec{x})\frac{\partial^2}{\partial u^2}K(\vec{x}) - \left(\frac{\partial^2}{\partial u \partial v}K(\vec{x})\right)^2\right) dz^2 + \\
&+2\left(\frac{\partial^2}{\partial u \partial w}K(\vec{x})\frac{\partial^2}{\partial u \partial v}K(\vec{x}) - \frac{\partial^2}{\partial u^2}K(\vec{x})\frac{\partial^2}{\partial v \partial w}K(\vec{x})\right) dzdy + \\
&+dxdu + dydv + dzdw \qquad (12)
\end{aligned}
$$

where $K(\vec{x}) = K(x, y, z, u, v, w)$ is arbitrary function.

The Ricci tensor R_{ij} of the metric (12) has a six components.
All equations

$$R_{ij} = 0$$

after the substitution

$$K(x, y, z, u, v, w) = \phi(y + v + x, z + w + x)$$

are reduced to the one equation

$$\Delta\psi(\xi, \rho) = 0$$

where $\xi = x + y + v,\ \rho = x + z + w$,

$$\psi(\xi,\ \rho) = \left(\frac{\partial^2}{\partial\xi^2}\phi(\xi, \rho)\right)\frac{\partial^2}{\partial\rho^2}\phi(\xi, \rho) - \left(\frac{\partial^2}{\partial\xi\partial\rho}\phi(\xi, \rho)\right)^2$$

and

$$\Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\rho^2}$$

is the Laplace operator.
Its solutions give the Ricci-flat examples of the metric (12).

The Beltrami parameters

To the investigation of the properties of the metrics (12) can be considered two invariant equations defined by the first

$$\Delta\psi = g^{ij}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^j}$$

and the second

$$\Box\psi = g^{ij}\left(\frac{\partial^2}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right)\psi$$

Beltrami parameters.

In particular case the second B-L equation $\Box\phi = 0$ after the substitution

$$\phi(\vec{x}) = f(\vec{x})$$

takes the form

$$\frac{\partial^2}{\partial u \partial x} f(\vec{x}) + \frac{\partial^2}{\partial w \partial z} f(\vec{x}) + \frac{\partial^2}{\partial v \partial y} f(\vec{x}) - 3 \left(\frac{\partial^2}{\partial y^2} f(\vec{x}) \right) \left(\frac{\partial^2}{\partial x^2} f(\vec{x}) \right) \frac{\partial^2}{\partial z^2} f(\vec{x}) +$$

$$+3 \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x}) \right)^2 \frac{\partial^2}{\partial y^2} f(\vec{x}) - 6 \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x}) \right) \left(\frac{\partial^2}{\partial y \partial z} f(\vec{x}) \right) \frac{\partial^2}{\partial x \partial y} f(\vec{x}) +$$

$$+3 \left(\frac{\partial^2}{\partial x \partial y} f(\vec{x}) \right)^2 \frac{\partial^2}{\partial z^2} f(\vec{x}) + 3 \left(\frac{\partial^2}{\partial y \partial z} f(\vec{x}) \right)^2 \frac{\partial^2}{\partial x^2} f(\vec{x}) = 0. \tag{13}$$

After the change of variables

$$f(\vec{x}) = f(x, y, z, u, v, w) = h(x + u, v + y, w + z) = h(\eta, \xi, \rho)$$

the equation (13) is reduced to the form

$$\Delta h(\eta,\xi,\rho) - 3\det\begin{bmatrix} \frac{\partial^2}{\partial\eta^2}h(\eta,\xi,\rho) & \frac{\partial^2}{\partial\eta\partial\xi}h(\eta,\xi,\rho) & \frac{\partial^2}{\partial\eta\partial\rho}h(\eta,\xi,\rho) \\ \frac{\partial^2}{\partial\eta\partial\xi}h(\eta,\xi,\rho) & \frac{\partial^2}{\partial\xi^2}h(\eta,\xi,\rho) & \frac{\partial^2}{\partial\rho\partial\xi}h(\eta,\xi,\rho) \\ \frac{\partial^2}{\partial\eta\partial\rho}h(\eta,\xi,\rho) & \frac{\partial^2}{\partial\rho\partial\xi}h(\eta,\xi,\rho) & \frac{\partial^2}{\partial\rho^2}h(\eta,\xi,\rho) \end{bmatrix} = 0, \tag{14}$$

where

$$\Delta = \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\rho^2}$$

is a three-dimensional Laplace operator.
The equation (14) is a famous Harvey-Lawson "Special Lagrangian" equation having an important applications in theory of Calabi-Yau manifolds and mirror symmetry.

Method of solution

To obtain particular solutions of the partial nonlinear differential equation

$$F(x, y, z, f_x, f_y, f_z, f_{xx}, f_{xy}, f_{xz}, f_{yy}, f_{yz}, f_{xxx}, f_{xyy}, f_{xxy}, ..) = 0 \quad (15)$$

can be applied a following approach.
We use the following parametric presentation of the functions and variables

$$f(x, y, z) \to u(x, t, z), \quad y \to v(x, t, z), \quad f_x \to u_x - \frac{u_t}{v_t} v_x,$$

$$f_z \to u_z - \frac{u_t}{v_t} v_z, \quad f_y \to \frac{u_t}{v_t}, \quad f_{yy} \to \frac{(\frac{u_t}{v_t})_t}{v_t}, \quad f_{xy} \to \frac{(u_x - \frac{u_t}{v_t} v_x)_t}{v_t}, ... \quad (16)$$

where variable t is considered as parameter.
Remark that conditions of the type

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}...$$

hold at the such type of presentation.

In result instead of equation (15) one get the relation between the new variables $u(x,t,z)$ and $v(x,t,z)$ and their partial derivatives

$$\Psi(u, v, u_x, u_z, u_t, v_x, v_z, v_t, ...) = 0. \tag{17}$$

This relation coincides with initial p.d.e at the condition $v(x,t,z) = t$ and takes more general form after presentation of the functions u, v in form $u(x,t,z) = F(\omega, \omega_t ...)$ and $v(x,t,z,s) = \Phi(\omega, \omega_t ...)$ with some function $\omega(x,t,z)$.
After the change variables and function with accordance of (16) the SLE-equation is transformed into the relation (17) and then after the substitution

$$u(x,t,z) = t\frac{\partial}{\partial t}\omega(x,t,z) - \omega(x,t,z), \quad v(x,t,z) = \frac{\partial}{\partial t}\omega(x,t,z). \tag{18}$$

it takes the form of p.d.e.

$$
\begin{gathered}
-3\left(\frac{\partial^2}{\partial x^2}\omega(x,t,z)\right)\frac{\partial^2}{\partial z^2}\omega(x,t,z)+3\left(\frac{\partial^2}{\partial x\partial z}\omega(x,t,z)\right)^2- \\
-\left(\frac{\partial^2}{\partial t^2}\omega(x,t,z)\right)\frac{\partial^2}{\partial x^2}\omega(x,t,z)+\left(\frac{\partial^2}{\partial t\partial x}\omega(x,t,z)\right)^2+ \\
+\left(\frac{\partial^2}{\partial t\partial z}\omega(x,t,z)\right)^2-\left(\frac{\partial^2}{\partial t^2}\omega(x,t,z)\right)\frac{\partial^2}{\partial z^2}\omega(x,t,z)+1=0.
\end{gathered}
\tag{19}
$$

The equation (19) looks as the sum of the 2D M-A equations with respect the variables (x, t), (x, z) and (t, z).

Let us consider some examples of its solutions.

1. To the (u, v)- transformation of the equation

$$\mu\,\Delta\left(f\right) + Hess\left(f\right) = 0 \tag{20}$$

The substitution into (19)

$$\omega\left(x, t, z\right) = A\left(x^2 + z^2, t\right)$$

lead to the equation on the function $A(\eta, t)$, $\eta = x^2 + z^2$

$$-4\,\mu\,\left(\frac{\partial^2}{\partial\eta\partial t}A\left(\eta, t\right)\right)^2\eta - 8\,\left(\frac{\partial^2}{\partial\eta^2}A\left(\eta, t\right)\right)\eta\,\frac{\partial}{\partial\eta}A\left(\eta, t\right) -$$

$$-4\,\left(\frac{\partial}{\partial\eta}A\left(\eta, t\right)\right)^2 + 4\,\mu\,\left(\frac{\partial^2}{\partial t^2}A\left(\eta, t\right)\right)\frac{\partial}{\partial\eta}A\left(\eta, t\right) - \mu +$$

$$+4\,\mu\,\left(\frac{\partial^2}{\partial t^2}A\left(\eta, t\right)\right)\frac{\partial^2}{\partial\eta^2}A\left(\eta, t\right) = 0.$$

Its particular solution is

$$A\left(\eta, t\right) = B\left(t\right) + \eta\,e^{kt},$$

$$B\left(t\right) = \frac{e^{kt}}{\mu\,k^2} + 1/4\,\frac{e^{-kt}}{k^2} + \_C1\,t + \_C2.$$

Now elimination of the parameter t from the system

$$y-\frac{\partial}{\partial t}\omega\left(x,t,z\right)=0,\quad f\left(x,y,z\right)-t\frac{\partial}{\partial t}\omega\left(x,t,z\right)+\omega\left(x,t,z\right)=0$$

lead to the solution of the equation (20)

$$f\left(x,y,z\right)=-\left(1-\ln\left(\frac{y\mu+\sqrt{T}}{1+\mu\,x^2+\mu\,z^2}\right)y\sqrt{T}\right)\left(y\mu+\sqrt{T}\right)^{-1}+$$

$$+\left(\ln\left(2\right)y^2\mu-\ln\left(\frac{y\mu+\sqrt{T}}{1+\mu\,x^2+\mu\,z^2}\right)y^2\mu\right)\left(y\mu+\sqrt{T}\right)^{-1}+$$

$$+\left(y^2\mu+\mu\,x^2\right)+\left(\ln\left(2\right)y\sqrt{T}+\mu\,z^2+y\sqrt{T}\right)\left(y\mu+\sqrt{T}\right)^{-1},$$

where

$$\mu\,\left(y^2\mu+1+\mu\,x^2+\mu\,z^2\right)=T$$

2. The substitution

$$\omega(x,t,z) = A(x+t,z)$$

into (19) lead to the equation on the function $A(eta,z)$, $\eta = x+z$

$$-\left(\frac{\partial^2}{\partial \eta^2}A(\eta,z)\right)\frac{\partial^2}{\partial z^2}A(\eta,z) - \mu\left(\frac{\partial^2}{\partial \eta \partial z}A(\eta,z)\right)^2 +$$

$$+\mu\left(\frac{\partial^2}{\partial \eta^2}A(\eta,z)\right)\frac{\partial^2}{\partial z^2}A(\eta,z) - \mu + \left(\frac{\partial^2}{\partial \eta \partial z}A(\eta,z)\right)^2 = 0$$

At the $\mu = -1/3$ we get the M-A equation

$$-\left(\frac{\partial^2}{\partial \eta^2}A(\eta,z)\right)\frac{\partial^2}{\partial z^2}A(\eta,z) + \left(\frac{\partial^2}{\partial \eta \partial z}A(\eta,z)\right)^2 + 1/4 = 0 \quad (21)$$

which after the (u,v)-transformation is reduced to the Laplace equation

$$4\frac{\partial^2}{\partial \eta^2}\theta(\eta,\xi) + \frac{\partial^2}{\partial \xi^2}\theta(\eta,\xi) = 0.$$

Its general solution

$$\theta(\eta,\xi) = M(\eta + 2I\xi) + N(\eta - 2I\xi)$$

Heavenly metrics and Y-M equation

We consider following generalization of Heavenly metrics

$$ {}^{8}ds^2 = dx\,du + dy\,dv + dz\,dw + A(x,y,z,p,q)du^2 + 2\,B(x,y,z,p,q)du\,dv + $$
$$ +2\,E(x,y,z,p,q)du\,dw + C(x,y,z,p,q)dv^2 + 2\,H(x,y,z,p,q)dv\,dw + $$
$$ +F(x,y,z,p,q)dw^2 + dp\,dq, \qquad (22) $$

having a following components

$$ A(\vec{x}) = \left(\frac{\partial^2}{\partial z^2} f(\vec{x})\right) \frac{\partial^2}{\partial y^2} f(\vec{x}) - \left(\frac{\partial^2}{\partial y \partial z} f(\vec{x})\right)^2, $$

$$ C(\vec{x}) = \left(\frac{\partial^2}{\partial x^2} f(\vec{x})\right) \frac{\partial^2}{\partial z^2} f(\vec{x}) - \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x})\right)^2, $$

$$ F(\vec{x}) = \left(\frac{\partial^2}{\partial x^2} f(\vec{x})\right) \frac{\partial^2}{\partial y^2} f(\vec{x}) - \left(\frac{\partial^2}{\partial x \partial y} f(\vec{x})\right)^2, $$

$$E(\vec{x}) = \left(\frac{\partial^2}{\partial y \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial x \partial y} f(\vec{x}) - \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial y^2} f(\vec{x}),$$

$$B(\vec{x}) = \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial y \partial z} f(\vec{x}) - \left(\frac{\partial^2}{\partial x \partial y} f(\vec{x})\right) \frac{\partial^2}{\partial z^2} f(\vec{x}),$$

$$H(\vec{x}) = \left(\frac{\partial^2}{\partial x \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial x \partial y} f(\vec{x}) - \left(\frac{\partial^2}{\partial y \partial z} f(\vec{x})\right) \frac{\partial^2}{\partial x^2} f(\vec{x}).$$

At these conditions from 21 components of the Ricci-tensor of the metric (22) only six components

$$R_{uu} \neq 0, \quad R_{uv} \neq 0, \quad R_{uw} \neq 0, \quad R_{vv} \neq 0, \quad R_{vw} \neq 0, \quad R_{ww} \neq 0.$$

are different from zero.

The equation

$$g^{ij}\left(\frac{\partial^2\Psi}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial\Psi}{\partial x^k}\right)=0,$$

defined by the Laplace-Beltrami operator of the metric (22) depends on the function $f(x,y,z,p,q)$ and after the substitution $f=\Psi$ takes the form

$$\frac{\partial^2}{\partial p\partial q}f(x,y,z,p,q)-$$

$$-3\begin{bmatrix} \frac{\partial^2}{\partial x^2}f(x,y,z,p,q) & \frac{\partial^2}{\partial x\partial y}f(x,y,z,p,q) & \frac{\partial^2}{\partial x\partial z}f(x,y,z,p,q) \\ \frac{\partial^2}{\partial x\partial y}f(x,y,z,p,q) & \frac{\partial^2}{\partial y^2}f(x,y,z,p,q) & \frac{\partial^2}{\partial y\partial z}f(x,y,z,p,q) \\ \frac{\partial^2}{\partial x\partial z}f(x,y,z,p,q) & \frac{\partial^2}{\partial y\partial z}f(x,y,z,p,q) & \frac{\partial^2}{\partial z^2}f(x,y,z,p,q) \end{bmatrix}=0. \tag{23}$$

The equation (23) is of Monge-Ampere type and after the substitution

$$f(x,y,z,p,q) = U(p,q)E(x,y,z)$$

takes the form

$$\frac{\frac{\partial^2}{\partial p \partial q}U(p,q)}{\left(U(p,q)\right)^3} - 3\,\frac{Hess(E(x,y,z))}{E(x,y,z)} = 0.$$

So it has particular solutions defined by the equations

$$3Hess(E) - \mu E = 0$$

and

$$\frac{\partial^2 U}{\partial p \partial q} - \mu U^3 = 0.$$

Both of equations have an important applications.
The second equation is reduction of the $SU(2)$ Yang-Mills equation and the first is a Monge-Ampere type of equation having applications in theory of Calabi-Yau manifolds.

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