Hodograph Solutions of Generalized Dispersionless KP Equation

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Outline

Introduction

The generalized dispersionless KP equation(or Manakov-Santini equation) is defined by [2006]

$$\begin{aligned} \frac{1}{3}u_{1xt} &= \frac{1}{4}u_{1yy} + (u_1u_{1x})_x + \frac{1}{2}v_{1x}u_{1xy} - \frac{1}{2}u_{1xx}v_{1y}, \\ \frac{1}{3}v_{1xt} &= \frac{1}{4}v_{1yy} + u_1v_{1xx} + \frac{1}{2}v_{1x}v_{1xy} - \frac{1}{2}v_{1xx}v_{1y}, \end{aligned} \tag{1}$$

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From the second equation, one can see that u_1 can be expressed as differential polynomial of v_1, After plugging it into the first equation, we can obtain the non-linear $fourth \quad order$ (2+1)-dimensional P.D.E. for v_1.

It is noticed that for $v_1 = 0$ reduction, the system reduces to the famous dKP equation

$$\frac{1}{3}u_{1xt} = \frac{1}{4}u_{1yy} + (u_1u_{1x})_x. \qquad (2)$$

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And $u_1 = 0$ reduction gives the equation associated with Einstein-Weyl space [2002-2004: Shabat, Alonso, Pavlov, Dunajski]

$$\frac{1}{3}v_{1xt} = \frac{1}{4}v_{1yy} + \frac{1}{2}v_{1x}v_{1xy} - \frac{1}{2}v_{1xx}v_{1y}. \tag{3}$$

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When $u_1 = \frac{v_{1y}}{2} = \frac{\phi_{yy}}{2}$, we can obtain the dispersionless discrete KP equation:

$$\frac{1}{3}\phi_{xt} = \frac{1}{4}\phi_{yy} + \frac{1}{4}\phi_{xy}^2$$

Remark: Einstein-Weyl Space[Weyl, 1918] Let $\mathcal{W}$ be a three dimensional complex manifold, with a torsion-free connection D and conformal metric [h]. We also call $\mathcal{W}$ a Weyl space if the null geodesic (light geodesic) of [h] are also geodesic for D. It is equivalent to

$$D_i h_{jk} = \nu_i h_{jk} \tag{4}$$

for some one form ν(electro-magnetic field). The Weyl connection G^k_{ij} of D can be expressed as

$$G^k_{ij} = \Gamma^k_{ij} - \frac{1}{2}(\delta^k_j \nu_i + \delta^k_i \nu_j - h_{ij} h^{kl} \nu_l).$$

The Ricci tensor W_{ij} and the scalar curvature W of D are related to the Ricci tensor R_{ij} and the scalar curvature of [h] by

$$W_{ij} = R_{ij} + \nabla_i \nu_j - \frac{1}{2}\nabla_j \nu_i + \frac{1}{4}\nu_i \nu_j + h_{ij}(\frac{1}{2}\nabla_k \nu^k - \frac{1}{4}\nu_k \nu^k),$$
$$W = h^{ij} W_{ij} = R + 2\nabla^k \nu_k - \frac{1}{2}\nu^k \nu_k.$$

In term of Riemann data the EW equations are

$$W_{(ij)} = \frac{W_{ij} + W_{ji}}{2} = \frac{1}{3} W h_{ij}.$$

or

$$\chi_{ij} = R_{ij} + \frac{1}{4}(\nabla_i \nu^j + \nabla_j \nu^i) + \frac{1}{4}\nu_i \nu_j - \frac{1}{3}(R + \frac{1}{2}\nabla^k \nu_k + \frac{1}{4}\nu^k \nu_k) h_{ij} = 0.$$

$$W_{ij} = R_{ij} + \nabla_i\nu_j - \frac{1}{2}\nabla_j\nu_i + \frac{1}{4}\nu_i\nu_j + h_{ij}(\frac{1}{2}\nabla_k\nu^k - \frac{1}{4}\nu_k\nu^k),$$
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Example[Dunajski, 2008] Let

$$h = (dy - v_x dt)^2 - 4(dx - (u - v_y)dt)dt,$$
$$\nu = -v_{xx}dy + (4u_x - 2v_{xy} + v_x v_{xx})dt$$

Then the EW equation reduces to the generalized dKP equation

Lax Representation

We define the Lax operator $\mathcal{L}$ and the Orlov operator $\mathcal{M}$ as

$$\mathcal{L} = p + \sum_{n=1}^{\infty} u_n(x) p^{-n}, \quad (5)$$

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + \sum_{n=1}^{\infty} v_n(x) \mathcal{L}^{-n}. \quad (6)$$

Then the generalized dKP hierarchy is constructed by the Lax-Sato equation [Bogdanov, Dryuma, Manokov, 2006]

$$\frac{\partial \psi}{\partial t_n} = A_n \frac{\partial \psi}{\partial x} - B_n \frac{\partial \psi}{\partial p}, \qquad \psi = \begin{pmatrix} \mathcal{L} \\ \mathcal{M} \end{pmatrix}, \tag{7}$$

where $A_n \equiv (J_0^{-1} \partial \mathcal{L}^n / \partial p)_+$, $B_n \equiv (J_0^{-1} \partial \mathcal{L}^n / \partial x)_+$.Here $(\cdots)_+$ denote respectively the projection on the polynomial part, and J_0 is defined by

$$\begin{aligned} J_0 &= \frac{\partial \mathcal{L}}{\partial p}\frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial p} = \frac{\partial \mathcal{L}}{\partial p}\left(\frac{\partial \mathcal{M}}{\partial x}\Big|_{\mathcal{L}\text{ fixed}}\right) \\ &= 1 + v_{1x} p^{-1} + (v_{2x} - u_1) p^{-2} + (v_{3x} - 2u_2 - 2v_{1x} u_1) p^{-3} + \cdots . \end{aligned}$$

For instance,

$$
\begin{aligned}
A_1 &= 1, \\
A_2 &= 2p - 2v_{1x}, \\
A_3 &= 3p^2 - 3v_{1x}p + 6u_1 + 3(v_{1x})^2 - 3v_{2x}, \\
A_4 &= 4p^3 - 4v_{1x}p^2 + (12u_1 + 4(v_{1x})^2 - 4v_{2x})p \\
&\quad +12u_2 - 4v_{3x} + 8v_{1x}v_{2x} - 4(v_{1x})^3 - 8u_1v_{1x}, \qquad (8)
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and

$$\begin{aligned} B_1 &= 0, \\ B_2 &= 2u_{1x}, \\ B_3 &= 3u_{1x}p - 3u_{1x}v_{1x} + 3u_{2x}, \\ B_4 &= 4u_{1x}p^2 + (4u_{2x} - 4u_{1x}v_{1x})p \\ &\quad +4u_{1x}(4u_1 + (v_{1x})^2 - v_{2x}) - 4u_{2x}v_{1x} + 4u_{3x}. \end{aligned} \tag{9}$$

We remark that since

$$\frac{\partial \mathcal{M}}{\partial t_n}\bigg|_{\mathcal{L}} + \frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial t_n} = A_n(p(\mathcal{L}))\left(\frac{\partial \mathcal{M}}{\partial x}\bigg|_{\mathcal{L}} + \frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial x}\right) - B_n(p(\mathcal{L}))\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial p},$$

after eliminating the Lax equation of $\mathcal{L}$, one has

$$\frac{\partial \mathcal{M}}{\partial t_n}\bigg|_{\mathcal{L}} = A_n(p(\mathcal{L}))\left(\frac{\partial \mathcal{M}}{\partial x}\bigg|_{\mathcal{L}}\right), \tag{10}$$

where $p(\mathcal{L})$ is the inverse of Lax operator $\mathcal{L}(p)$, i.e.,

$$p(\mathcal{L}) = \mathcal{L} - \sum_{n=1}^{\infty} \tilde{u}_n \mathcal{L}^{-n}$$

For example,

$$\tilde{u}_1 = u_1,\ \tilde{u}_2 = u_2,\ \tilde{u}_3 = u_1^2 + u_3,\ \tilde{u}_4 = 3u_1u_2 + u_4.$$

Now, from the the evolution of $\mathcal{L}, \mathcal{M}$ with respect to $t_2 = y$, one knows

$$\frac{1}{2}\frac{\partial \mathcal{L}}{\partial y} = (p - v_{1x})\frac{\partial \mathcal{L}}{\partial x} - u_{1x}\frac{\partial \mathcal{L}}{\partial p}, \tag{11}$$

$$\frac{1}{2}\frac{\partial \mathcal{M}}{\partial y} = (p - v_{1x})\frac{\partial \mathcal{M}}{\partial x}. \tag{12}$$

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And then

$$\frac{1}{2}u_{1y} = u_{2x} - v_{1x}u_{1x}, \quad (13)$$

$$\frac{1}{2}u_{2y} = u_{3x} - v_{1x}u_{2x} + u_1 u_{1x}, \quad (14)$$

$$v_{2x} = u_1 + v_{1x}^2 + \frac{1}{2}v_{1y}, \quad (15)$$

$$v_{3x} = u_2 + \frac{1}{2}v_{2y} + u_1 v_{1x} + v_{1x}v_{2x}. \quad (16)$$

Similarly, the the evolution of $\mathcal{L}, \mathcal{M}$ with respect to $t_3 = t$ gives

$$\begin{aligned}
\frac{1}{3}\frac{\partial \mathcal{L}}{\partial t} &= \left(p^2 - v_{1x}p + u_1 - \frac{1}{2}v_{1y}\right)\frac{\partial \mathcal{L}}{\partial x} - \left(u_{1x}p + \frac{1}{2}u_{1y}\right)\frac{\partial \mathcal{L}}{\partial p}, \\
\frac{1}{3}\frac{\partial \mathcal{M}}{\partial t} &= \left(p^2 - v_{1x}p + u_1 - \frac{1}{2}v_{1y}\right)\frac{\partial \mathcal{M}}{\partial x}.
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\frac{1}{3}\frac{\partial \mathcal{M}}{\partial t} &= \left(p^2 - v_{1x}p + u_1 - \frac{1}{2}v_{1y}\right)\frac{\partial \mathcal{M}}{\partial x}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{3}u_{1t} &= u_{3x} - v_{1x}u_{2x} + (u_1 - \frac{1}{2}v_{1y})u_{1x} + u_1 u_{1x}, \\
&= \frac{1}{2}u_{2y} - \frac{1}{2}u_{1x}v_{1y} + u_1 u_{1x}, \\
\frac{1}{3}v_{1t} &= -u_2 + \frac{1}{2}v_{2y} + u_1 v_{1x} - \frac{1}{2}v_{1x}v_{1y}, \qquad (17)
\end{aligned}$$

From these equations, we can elinminate the u_{2x} and v_{2x} and obtain the generalized dKP equation

$$\begin{aligned}\frac{1}{3}u_{1xt} &= \frac{1}{4}u_{1yy} + (u_1 u_{1x})_x + \frac{1}{2}v_{1x}u_{1xy} - \frac{1}{2}u_{1xx}v_{1y},\\ \frac{1}{3}v_{1xt} &= \frac{1}{4}v_{1yy} + u_1 v_{1xx} + \frac{1}{2}v_{1x}v_{1xy} - \frac{1}{2}v_{1xx}v_{1y},\end{aligned}$$

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\end{aligned}$$

Remark: The generalized dKP equation can also be obtained as the compatibility condition of the Hamilton-Jacobi type equation [Bogdanov, Chang, Chen, 2008]

$$\left.\frac{\partial p(\mathcal{L})}{\partial t_n}\right|_{\mathcal{L}\ \text{fixed}} = A_n(p(\mathcal{L}))\left.\frac{\partial p(\mathcal{L})}{\partial x}\right|_{\mathcal{L}\ \text{fixed}} + B_n(p(\mathcal{L})). \qquad (18)$$

Finite Dimensional Reduction

Let us introduce the generating functions

$$Q := (\partial_x \mathcal{M}\big|_{\mathcal{L}})^{-1}. \tag{19}$$

Then equation (10) is equivalent to

$$\partial_{t_n} Q = A_n(\partial_x Q) - (\partial_x A_n) Q := \langle A_n, Q \rangle. \tag{20}$$

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Next, we consider the reduction of (20) by the solutions $Q = Q(p(\mathcal{L}), \boldsymbol{V})$, where $\boldsymbol{V} = (V_1, \ldots, V_m)$. Then using the equation (18), the equation (20) becomes

$$\partial_{t_n} Q\big|_p = A_n(\partial_x Q\big|_p) - (\partial_x A_n) Q - B_n(\partial_p Q). \tag{21}$$

For simplicity, letting $v_2 = 0$ and after some trials and errors, we assume

$$Q = \frac{p}{p - v}, \tag{22}$$

where $V_1 := v = -v_{1,x}$. Substituting (22) into (21) we derive

$$\partial_{t_n} v = A_n(\partial_x v) - (\partial_x A_n)(p - v) + \frac{v}{p} B_n.$$

Evaluated at $p = v$, the root of Q^{-1} gives the evolution equations of v

$$\partial_{t_n} v = A_n(p = v)\partial_x v + B_n(p = v). \tag{23}$$

The first three evolution equations are given as

$$\begin{aligned}
\partial_{t_2} v &= 4vv_x + 2u_{1,x}, \\
\partial_{t_3} v &= 9v^2 v_x + 6(vu_1)_x + 3u_{2,x},
\end{aligned}$$

On the other hand, from the Eq.(18) for $n = 2, 3$, one has

$$\begin{aligned}\partial_{t_2} p &= 2(p+v)p_x + 2u_{1,x}, \\ \partial_{t_3} p &= 3(p^2 + vp + 2u_1 + v^2)p_x + 3(p+v)u_{1,x} + 3u_{2,x}.\end{aligned}$$

Then the compatibility condition $\partial_{t_2}\partial_{t_3} p = \partial_{t_3}\partial_{t_2} p$ with independent variables $p^0, p, p_x, pp_x,$ implies

$$\begin{aligned}&6vu_{2,xx} + 6vv_x u_{1,x} - 3u_{2,xt_2} - 6(u_{1,x})^2 + \\ &\quad -12u_1 u_{1,xx} - 3v_y u_{1,x} - 3vu_{1,xt_2} + 2u_{1,xt_3} = 0, \qquad (24)\\ &2u_{2,xx} + 2(vu_{1,x})_x - u_{1,xt_2} = 0, \qquad (25)\\ &3u_{2,x} + v_{t_3} + 6vu_{1,x} + 3v^2 v_x - 3vv_{t_2} - 6u_1 v_x - 3u_{1,t_2} \qquad (26)\\ &= 0\\ &2u_{1,x} + 4vv_x - v_{t_2} = 0. \qquad (27)\end{aligned}$$

By equations (25), (27) we obtain

$$\partial_{t_2} u_1 = 2vu_{1,x} + 2u_{2,x}, \tag{28}$$
$$\partial_{t_2} v = 4vv_x + 2u_{1,x}. \tag{29}$$

Substituting into (24), (26) we have also

$$\partial_{t_3} u_1 = (3/2)u_{2,y} + 6u_1u_{1,x} + 3v^2u_{1,x}, \tag{30}$$
$$\partial_{t_3} v = 9v^2v_x + 3u_{2,x} + 6(vu_1)_x. \tag{31}$$

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It is easy to see that Eqs.(29) and (31) are nothing but the evolution equations of v. Therefore one shows that the compatibility $\partial_{t_2}\partial_{t_3} p = \partial_{t_3}\partial_{t_2} p$ implies $\partial_{t_2}\partial_{t_3} v = \partial_{t_3}\partial_{t_2} v$. Then the reduction (22) is admissible with the general expansion of Lax operator (5) to have commuting flows , at least, up to t_3.

We shall use the nth-KdV reduction of Lax operator arising from the dKP hierarchy, i.e.,

$$\mathcal{L}^{n+1} = p^{n+1} + w_1 p^{n-1} + w_2 p^{n-2} + \cdots + w_{n-1} + w_n,$$

to construct finite-dimensional hydrodynamical systems in two- and three variable. Then the hodograph method can be used. Notice that they are different from that of ordinary dKdV and dBoussinesq equations. In generalized dKP equation, a new variable v is involved.

Remark: Consider the waterbag-type reduction of the generalized dKP hierarchy represented by

$$\mathcal{L} = p + \sum_{i=1}^{N} \epsilon_i \log(p - U_i), \quad (32)$$

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + \sum_{i=1}^{M} \delta_i \log(p - V_i), \quad (33)$$

where ϵ_i and δ_i are assumed to satisfy

$$\sum_{i=1}^{N} \epsilon_i = \sum_{i=1}^{M} \delta_i = 0. \quad (34)$$

It is shown that it can be reduced to the non-homogeneous Riemann invariant form [Bogdanov, Chang, Chen, 2008]. But it is still unknown to solve them.

Hodograph Method

In 1988, Gibbons and Kodama developed an systematic way to generalize the hodograph transformation to solve hydrodynamic system with enough symmetries. This method reduces the hydrodynamic system to a linear one by the relations of commuting flows: $\partial^2 u_i/\partial t_n \partial t_m = \partial^2 u_i/\partial t_m \partial t_n$.

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The Quasi-linear system is defined by

$$\partial_{t_l} u_i = \sum_{j=1}^{N} a_{ij}^l \partial_x u_j, \quad i, l = 1, 2, \ldots, N. \tag{35}$$

where a_{ij}^l are functions of $(u_1, \ldots, u_N)$ and $a_{ij}^1 = \delta_{ij}$. Then the generalized hodograph transformation means the interchange of the dependent and independent variables: $(t_1(\vec{u}), \ldots, t_N(\vec{u}))$.

The eq. (35) can be reduced to the following non-linear PDEs called hodograph equations $(t_1, \ldots, t_N)$: [Gibbons and Kodama, 1988]

$$f_j^l = \sum_{k=1}^{N} a_{jk}^l f_k^1, \quad l = 2, 3, \ldots, N, \tag{36}$$

where f_j^l are cofactors of the Jacobian $J = \partial(t_1, \ldots, t_N)/\partial(u_1, \ldots, u_N)$ for which $J \neq 0$. It has been shown that f_j^1 can be determined by a linear system of the defining equations

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$$\sum_{j=1}^{N} \partial_j f_j^l = \sum_{k=1}^{N}\sum_{j=1}^{N} \partial_j (a_{jk}^l f_k^1) = 0, \quad l = 1, \ldots, N. \tag{37}$$

where $\partial_j := \partial/\partial u_j$.

Using the fact

$$\sum_{j=1}^{N}(\partial_j t_n)f_j^l = \delta_{ln}\frac{\partial(t_1,\ldots,t_N)}{\partial(u_1,\ldots,u_N)}$$

and the hodograph equation (36), we have

$$\sum_{k=1}^{N}\left[\delta_{ln}\partial_k t_1 - \sum_{j=1}^{N}(\partial_j t_n)a_{jk}^l\right] f_k^1 = 0.$$

From this equation, one obtains the following linear system as the dual equations of the hodograph equation

$$\delta_{ln}\partial_k t_1 - \sum_{j=1}^{N}(\partial_j t_n)a^l_{jk} = \sum_{r=2}^{N}\phi^l_{nr}\partial_k t_r, \quad l=2,\ldots,N, \quad n,k=1,\ldots,N, \tag{38}$$

where $\phi^l_{nr} = \phi^l_{nr}(u_1,\ldots,u_N)$ are functions to be determined by a particular solution. And then by the dual system (38), we can construct polynomial solutions of higher scaling weights without using the hodograph equation (36)(see below).

Exact Solutions

♣ (1,1)-reduction: dKdV type

In this case $\mathcal{L}^2 = p^2 + u$, using the first few A_n and B_n

$$A_1 = 1, \quad A_2 = 2(p+v), \quad A_3 = 3(p^2 + vp + u + v^2), \tag{39}$$
$$B_1 = 0, \quad B_2 = u_x, \quad B_3 = \frac{3}{2}(p+v)u_x,$$

and from Eqs.(7) for $\mathcal{L}$ and (23), we obtain the first few nontrivial equations

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and from Eqs.(7) for $\mathcal{L}$ and (23), we obtain the first few nontrivial equations

$$\partial_y u = 2vu_x, \quad \partial_t u = 3uu_x + 3v^2 u_x, \quad \partial_{t_4} u = 6vuu_x + 4v^3 u_x,$$
$$\partial_y v = 4vv_x + u_x, \quad \partial_t v = 9v^2 v_x + 3(uv)_x,$$
$$\partial_{t_4} v = 16v^3 v_x + 6(v^2 u)_x + 3uu_x, \tag{40}$$

where $t_1 = x, t_2 = y, t_3 = t$. They satisfy not only the compatibility conditions $\partial_y \partial_t u = \partial_t \partial_y u$ and $\partial_y \partial_t v = \partial_t \partial_y v$, but also the commuting flows for all hierarchy.

On the other hand, by the relation (19), we derive some of the conserved quantities

$$v_{1,x} = -v, \quad v_{3,x} = -\frac{1}{2}uv, \quad v_{5,x} = -\frac{3}{8}u^2v, \quad v_{7,x} = -\frac{5}{16}u^3v, \quad \ldots$$
$$v_{n,x} = 0, \quad n \in \text{even}.$$

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Indeed, we list some conservation laws as following:

$$\partial_y(v_{1,x}) = -v_y = -\partial_x(u + 2v^2), \tag{41}$$

$$\partial_t(v_{1,x}) = -v_t = -3\partial_x(uv + v^3), \tag{42}$$

$$\begin{aligned}
\partial_y(v_{3,x}) &= -\frac{1}{2}(uv)_y = -\frac{1}{4}\partial_x(u^2 + 4uv^2), \\
\partial_t(v_{3,x}) &= -\frac{1}{2}(uv)_t = -\frac{3}{2}\partial_x(u^2v + uv^3), \\
\partial_y(v_{5,x}) &= -\frac{3}{8}(u^2v)_y = -\frac{1}{8}\partial_x(u^3 + 6u^2v^2), \\
\partial_t(v_{5,x}) &= -\frac{3}{8}(u^2v)_t = -\frac{9}{8}\partial_x(u^3v + u^2v^3).
\end{aligned}$$

The system by (40) is read as the quasi-linear form

$$\begin{pmatrix} u \\ v \end{pmatrix}_y = a^2 \begin{pmatrix} u \\ v \end{pmatrix}_x, \quad (43)$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = a^3 \begin{pmatrix} u \\ v \end{pmatrix}_x, \quad (44)$$

where the matrix

$$(a^2)_{2\times 2} = \begin{pmatrix} 2v & 0 \\ 1 & 4v \end{pmatrix}, \qquad (a^3)_{2\times 2} = \begin{pmatrix} 3u + 3v^2 & 0 \\ 3v & 3u + 9v^2 \end{pmatrix}.$$

The defining equations (37) of $f_1^1 = \partial_v y$ and $f_2^1 = -\partial_u y$ $(\partial_1 := \partial_u, \partial_2 := \partial_v)$ are

$$\begin{pmatrix} \partial_u & \partial_v \\ 2v\partial_u + \partial_v & 4\partial_v v \end{pmatrix} \begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix} = 0, \quad (45)$$

which admit the polynomial type solutions of the following form with scaling weights $[u] = 2$ and $[v] = 1$:

$$f_1^1 = \sum_{2l_1+l_2=K-1} \alpha_{l_1,l_2} u^{l_1} v^{l_2}, \qquad f_2^1 = \sum_{2l_1+l_2=K-2} \beta_{l_1,l_2} u^{l_1} v^{l_2},$$

where α_{l_1,l_2} and β_{l_1,l_2} are some constants to be determined and $K = 1, 2, \ldots$

For instance, for $K = 1$ we have $f_1^1 = \alpha, f_2^1 = \beta$. Substituting them into (45) and the hodograph equation (36), one yields

$$y = \alpha v, \quad x = \alpha(u - v^2), \tag{46}$$

where α is now an arbitrary constant. Furthermore, it is easy to show that based on (46) the correction terms ϕ_{n2}^2 of the dual linear equation (38) are determined : $\phi_{12}^2 = 8v^2, \phi_{22}^2 = -6v$. Thus, equation (38) provides

$$\begin{aligned} 0 &= \sum_{j=1}^{2} (\partial_j x) a_{jk}^2 + 8v^2\, \partial_k y, \\ \partial_k x &= \sum_{j=1}^{2} (\partial_j y) a_{jk}^2 - 6v\, \partial_k y, \end{aligned} \tag{47}$$

which allows a direct way to solve the higher weight of polynomial solutions.

In practice, let us look for the next solutions, $K = 2$. We have $x = c_1uv + c_2v^3$ and $y = c_3u + c_4v^2$. Making use of (47) we solve $c_1 = 0,\ c_3 = -3c_2/8,\ c_4 = -3c_2/4$, thus

$$x = c_2v^3 \qquad y = -\frac{3}{8}c_2(u + 2v^2),$$

where c_2 is an arbitrary constant. Similar procedures can be done for finding other polynomial-type solutions, We list first few of them with fixed scaling constant:

Table: Polynomial solutions of the dKdV type reduction.

K	f_1^1	f_2^1	x	y
1	1	0	$u - v^2$	v
2	$12v$	-3	$-8v^3$	$3u + 6v^2$
3	$u + 3v^2$	$-v$	$\frac{1}{2}u^2 - uv^2 - \frac{3}{2}v^4$	$uv + v^3$
4	$uv + \frac{4}{3}v^3$	$-\frac{1}{4}u - \frac{1}{2}v^2$	$-\frac{2}{3}uv^3 - \frac{8}{15}v^5$	$\frac{1}{8}u^2 + \frac{1}{2}uv^2 + \frac{1}{3}v^4$

We remark here that the above solutions of x and y can be treated as the initial value at $t = 0$. In the present system, the remaining task is finding the exact solutions in 2+1 dimensions (x, y, t).

We remark here that the above solutions of x and y can be treated as the initial value at $t = 0$. In the present system, the remaining task is finding the exact solutions in 2+1 dimensions (x, y, t).
It was stated by that the dependence of the time variable t may be found according to the relation of a^2 and a^3[Y. Kodama, 1988]

$$a^3 = \frac{3}{4}(a^2)^2 - \frac{3}{2}va^2 + 3(u + v^2)I, \tag{48}$$

where I is the 2×2 identity matrix. After changing $(x, y, t) \to (u, v, t)$ with the dependent variables $x = x(u, v, t)$ and $y = y(u, v, t)$, the hodograph equation corresponding to t-flow (44) (in addition to (36)) is given by

$$\begin{pmatrix} \partial(x,y)/\partial(v,t) \\ -\partial(x,y)/\partial(u,t) \end{pmatrix} = a^3 \begin{pmatrix} \partial_v y \\ -\partial_u y \end{pmatrix},$$

where $\partial(x,y)/\partial(v,t)$ and $\partial(x,y)/\partial(u,t)$ are the Jacobian of (x,y) with respect to (v,t) and (u,t), respectively.

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where $\partial(x,y)/\partial(v,t)$ and $\partial(x,y)/\partial(u,t)$ are the Jacobian of (x,y) with respect to (v,t) and (u,t), respectively. One can show that, combining the above and (36) with the relation (48) and requiring that $\partial_u y$ and $\partial_v y$ are independent variables, the implicit hodograph equations have the form (string equation)

$$\begin{aligned} x + \left(3(u+v^2) - \frac{3}{4}\det(a^2)\right)t &= t_1^0, \\ y + \left(\frac{3}{4}\mathrm{tr}(a^2) - \frac{3}{2}v\right)t &= t_2^0, \end{aligned} \qquad (49)$$

where $t_1^0 = t_1^0(u,v)$ and $t_2^0 = t_2^0(u,v)$ are initial values at $t = 0$ and can be found in Table 1.

Choose, for example, $K = 1$ and then equation (49) becomes

$$\begin{aligned} x + 3(u - v^2)t &= u - v^2, \\ y + 3vt &= v. \end{aligned} \tag{50}$$

We solve the hodograph solution as

$$u(x, y, t) = \frac{x - 3xt + y^2}{9t^2 - 6t + 1}, \qquad v(x, y, t) = \frac{-y}{3t - 1}.$$

One can verify that $u(x, y, t)$ and $v(x, y, t)$ satisfy (43) and (44).

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One can verify that $u(x, y, t)$ and $v(x, y, t)$ satisfy (43) and (44). Now using $u_1 = u/2$, $v_{1,x} = -v$ and the conservation equations (41), (42), i.e., $v_{1,y} = -(u + 2v^2), v_{1,t} = -3(uv + v^3)$, we obtain a solution of the Manakov-Santini equation

$$u_1(x, y, t) = \frac{1}{2}\frac{x - 3xt + y^2}{(3t - 1)^2}, \qquad v_1(x, y, t) = \frac{-y(x - 3xt + y^2)}{(3t - 1)^2}.$$

♣ (2,1)-reduction: dBoussinesq type
In this case, $\mathcal{L}^3 = p^3 + up + w$, comparing to (5) we have $u_1 = u/3, u_2 = w/3, u_3 = -u^2/9, u_4 = -2uw/9$, etc. Some of A_n and B_n are given by

$$A_1 = 1, \quad A_2 = 2(p+v), \quad A_3 = 3(p^2 + vp + \frac{2}{3}u + v^2),$$
$$B_1 = 0, \quad B_2 = \frac{2}{3}u_x, \quad B_3 = (p+v)u_x + w_x$$

Then from Eqs.(7) for $\mathcal{L}$ and (23) we derive the first few nontrivial equations

$$
\begin{aligned}
\partial_y u &= 2vu_x + 2w_x, \\
\partial_t u &= 3v^2u_x + uu_x + 3vw_x, \\
\partial_{t_4} u &= 4(v^3 + (2/3)vu + w)u_x + 4((2/3)u + v^2)w_x, \\
\partial_y w &= 2vw_x - \frac{2}{3}uu_x, \\
\partial_t w &= -vuu_x + 3v^2w_x + uw_x, \\
\partial_{t_4} w &= -4(v^2u/3 + (2/9)u^2)u_x + 4(v^3 + (2/3)vu + w)w_x, \\
\partial_y v &= 4vv_x + \frac{2}{3}u_x, \\
\partial_t v &= 9v^2v_x + 2(uv)_x + w_x, \\
\partial_{t_4} v &= 4(v^2 + (2/9)u)u_x \\
&+ (8/3)vw_x + 4(4v^3 + 2vu + w)v_x. \qquad (51)
\end{aligned}
$$

where $t_1 = x, t_2 = y, t_3 = t$.

We remark they are not compatible for t_4-flow; however, this does not affect the construction of the exact solutions in 2+1 dimensions: (x, y, t)
Also, the relation (19) gives that

$$v_{1,x} = -v, \quad v_{2,x} = 0, \quad v_{3,x} = -\frac{1}{3}uv, \quad v_{4,x} = -\frac{1}{3}wv,$$
$$v_{5,x} = -\frac{1}{9}u^2v, \quad v_{6,x} = -\frac{1}{3}uwv,$$

etc. In particular, $v_{1,x}$ satisfies

$$\partial_y(v_{1,x}) = -2\partial_x(u/3 + v^2), \qquad \partial_t(v_{1,x}) = -\partial_x(2uv + w + 3v^3). \tag{52}$$

We see that the system (51) can be written by

$$\begin{pmatrix} u \\ w \\ v \end{pmatrix}_y = \begin{pmatrix} 2v & 2 & 0 \\ -2u/3 & 2v & 0 \\ 2/3 & 0 & 4v \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x ,$$

$$\begin{pmatrix} u \\ w \\ v \end{pmatrix}_t = \begin{pmatrix} 3v^2+u & 3v & 0 \\ -uv & 3v^2+u & 0 \\ 2v & 1 & 9v^2+2u \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x .$$

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$$\begin{pmatrix} u \\ w \\ v \end{pmatrix}_y = \begin{pmatrix} 2v & 2 & 0 \\ -2u/3 & 2v & 0 \\ 2/3 & 0 & 4v \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x ,$$

$$\begin{pmatrix} u \\ w \\ v \end{pmatrix}_t = \begin{pmatrix} 3v^2+u & 3v & 0 \\ -uv & 3v^2+u & 0 \\ 2v & 1 & 9v^2+2u \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x .$$

The defining equations(37) for $f_1^1 = (\partial_w y)(\partial_v t) - (\partial_v y)(\partial_w t)$, $f_2^1 = -(\partial_u y)(\partial_v t) + (\partial_v y)(\partial_u t)$ and $f_3^1 = (\partial_u y)(\partial_w t) - (\partial_w y)(\partial_u t)$ $(\partial_1 := \partial_u, \partial_2 := \partial_w, \partial_3 := \partial_v)$ are

$$\begin{pmatrix} \partial_u & \partial_w & \partial_v \\ 2v\partial_u - \frac{2}{3}u\partial_w + \frac{2}{3}\partial_v & 2\partial_u + 2v\partial_w & 4\partial_v v \\ \partial_u u + 3v^2\partial_u - uv\partial_w + 2\partial_v v & 3v\partial_u + (u+3v^2)\partial_w + \partial_v & 2u\partial_v + 9\partial_v \end{pmatrix} \quad (53)$$

$$\begin{pmatrix} \partial_u & \partial_w & \partial_v \\ 2v\partial_u - \frac{2}{3}u\partial_w + \frac{2}{3}\partial_v & 2\partial_u + 2v\partial_w & 4\partial_v v \\ \partial_u u + 3v^2\partial_u - uv\partial_w + 2\partial_v v & 3v\partial_u + (u+3v^2)\partial_w + \partial_v & 2u\partial_v + 9\partial_v \end{pmatrix} \quad (53)$$

The polynomial type solutions admit the following form with scaling weights $[u] = 2, [w] = 3, [v] = 1$:

$$\begin{aligned} f_1^1 &= \sum_{2l_1+3l_2+l_3=2K-3} \alpha_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3}, \\ f_2^1 &= \sum_{2l_1+3l_2+l_3=2K-2} \beta_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3}, \\ f_3^1 &= \sum_{2l_1+3l_2+l_3=2K-4} \gamma_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3}, \end{aligned}$$

where $\alpha_{l_1,l_2,l_3}, \beta_{l_1,l_2,l_3}$ and γ_{l_1,l_2,l_3} are constants to be determined and $K = 1, 2, \ldots$.

To begin with, for $K = 1$ we have $f_1^1 = \alpha, f_2^1 = \beta$ and $f_3^1 = \gamma$. Plugging them into (53) and yielding $\alpha = \gamma = 0$, then by the definitions of f_k^1 and noticing that the scaling weights of u, w and v, one has

$$y = c_1\, u + c_2\, v^2, \qquad t = c_3\, v,$$

where c_1, c_2, c_3 are constants with $c_1 c_3 = -\beta$. Now substituting y and t into the hodograph equation (36), for $l = 2, 3$, one finds the polynomial solutions as

$$x = -2c_1(uv - w), \qquad y = c_1(u - 3v^2/2), \qquad t = 2c_1 v, \quad (54)$$

where c_1 is an arbitrary constant.

Similarly, in this case the dual linear equation (given by (38))

$$\begin{aligned}\delta_{2n}\partial_k x &= \sum_{j=1}^{3}(\partial_j t_n)a_{jk}^2 + \sum_{r=2}^{3}\phi_{nr}^2\partial_k t_r,\\ \delta_{3n}\partial_k x &= \sum_{j=1}^{3}(\partial_j t_n)a_{jk}^3 + \sum_{r=2}^{3}\phi_{nr}^3\partial_k t_r, \quad n,k=1,2,3,\end{aligned} \qquad (55)$$

with unknown functions ϕ_{nr}^2, ϕ_{nr}^3 can be found by using the simplest one (54), i.e.,

$$\begin{aligned}&\phi_{12}^2 = 4v^2 + (8/3)u, \quad \phi_{13}^2 = 6v^3 + 8uv, \quad \phi_{22}^2 = -2v,\\ &\phi_{23}^2 = 3v^2 - u, \phi_{32}^2 = -4/3, \quad \phi_{33}^2 = -6v,\\ &\phi_{12}^3 = 6v^3 + 8uv, \quad \phi_{13}^3 = 9v^4 + 21uv^2 + 2u^2, \quad \phi_{22}^3 = 3v^2 - u,\\ &\phi_{23}^3 = 18v^3 + (3/2)uv, \quad \phi_{32}^3 = -6v, \quad \phi_{33}^3 = -9v^2 - 3u.\end{aligned}$$

So that (55) provide useful formulas for determining the polynomial-type solutions in higher weights.

For example, we derive the next set of polynomial solution for $K = 2$. The weights for x, y and t are now $4, 3$ and 2, respectively and admit the following general expressions

$$
\begin{aligned}
x &= c_1u^2 + c_2uv^2 + c_3wv + c_4v^4, \\
y &= c_5uv + c_6w + c_7v^3, \\
t &= c_8u + c_9v^2.
\end{aligned}
$$

Then substituting the above into (55) we find

For example, we derive the next set of polynomial solution for $K = 2$. The weights for x, y and t are now $4, 3$ and 2, respectively and admit the following general expressions

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Then substituting the above into (55) we find $c_2 = c_7 = -c_9 = 3c_1,\ c_3 = c_4 = c_5 = 0,\ c_6 = -3c_1/2,\ c_8 = -c_1$. Therefore, we have

$$x = c_1(u^2 + 3uv^2), \quad y = -\frac{3}{2}c_1(w - 2v^3), \quad t = -c_1(u + 3v^3),$$

where c_1 is an arbitrary constant.

To find the (2+1)-dimensional equations involving (x, y, t) that satisfy (51), we choose, for example, the expression of (54) and obtain the explicit hodograph solution

$$
\begin{aligned}
u(x,y,t) &= \frac{1}{c_1}y + \frac{3}{8c_1^2}t^2, \\
w(x,y,t) &= \frac{1}{2c_1}x + \frac{1}{2c_1^2}yt + \frac{3}{16c_1^3}t^3, \\
v(x,y,t) &= \frac{1}{2c_1}t.
\end{aligned}
$$

Finally, by the relations $u_1 = u/3$, $v_{1,x} = -v$ and (52), one can easily solve the solution satisfying the Manakov-Santini equation as

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$$\begin{aligned} u_1(x,y,t) &= \frac{1}{3c_1}y + \frac{1}{8c_1^2}t^2, \\ v_1(x,y,t) &= -\frac{1}{2c_1}xt - \frac{1}{3c_1}y^2 - \frac{3}{4c_1^2}yt^2 - \frac{15}{64c_1^3}t^4. \end{aligned}$$