On Dissipation Function of Ocean Waves due to Whitecapping.

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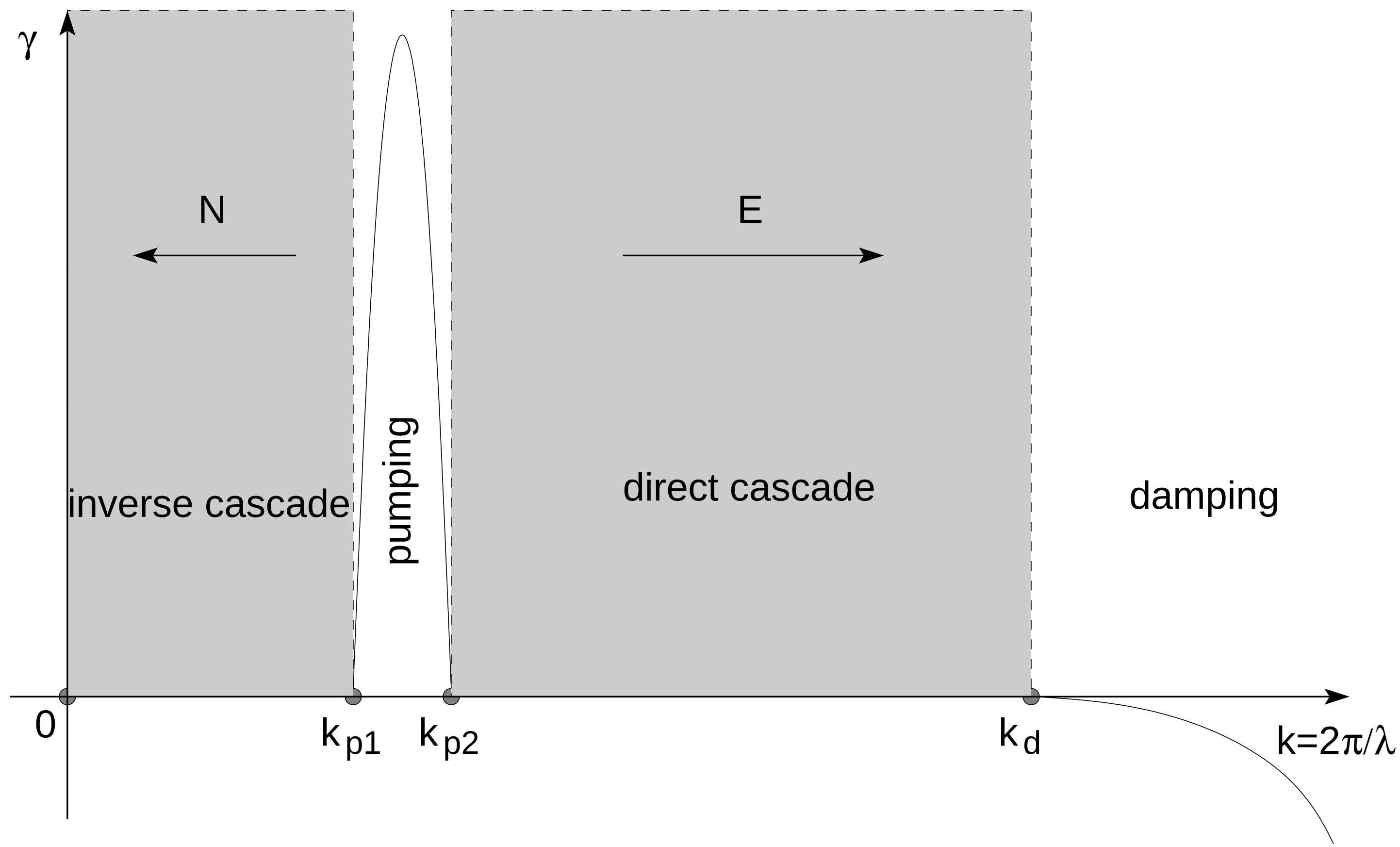
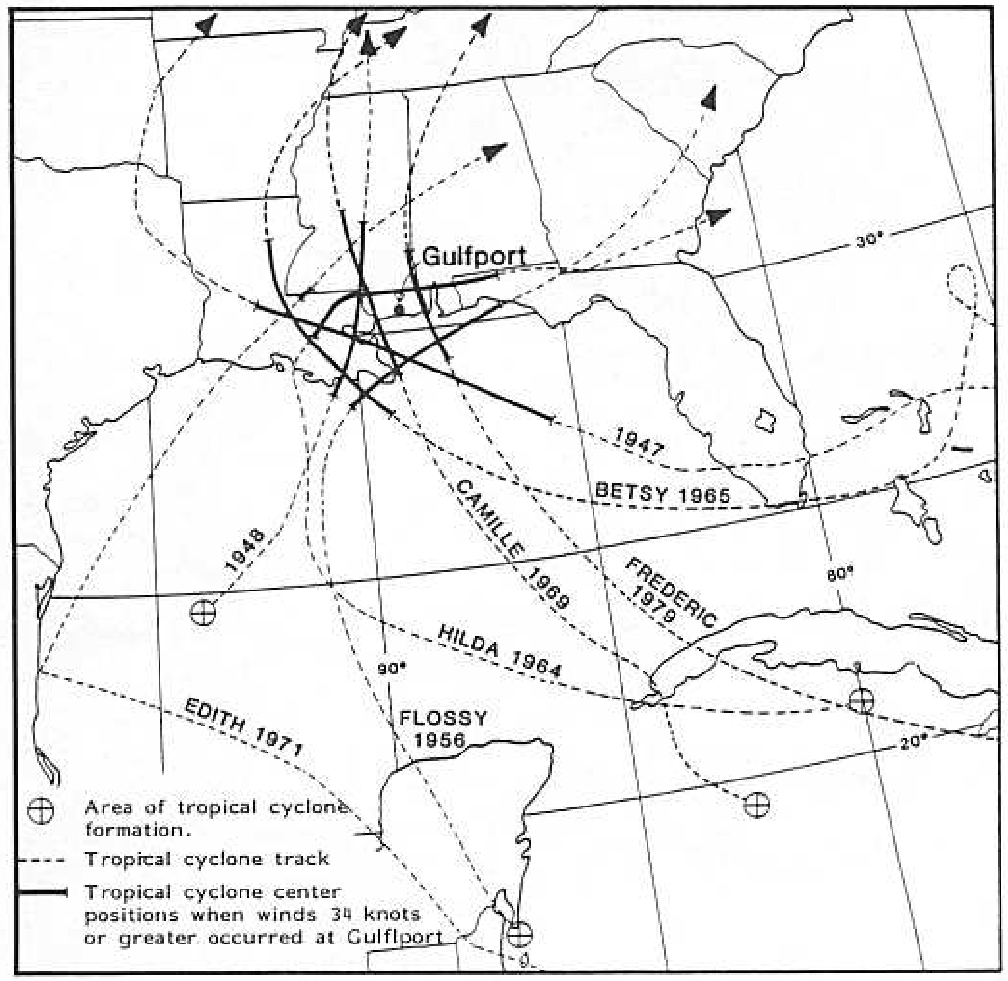
August 4, 2009

Waves forecasting.

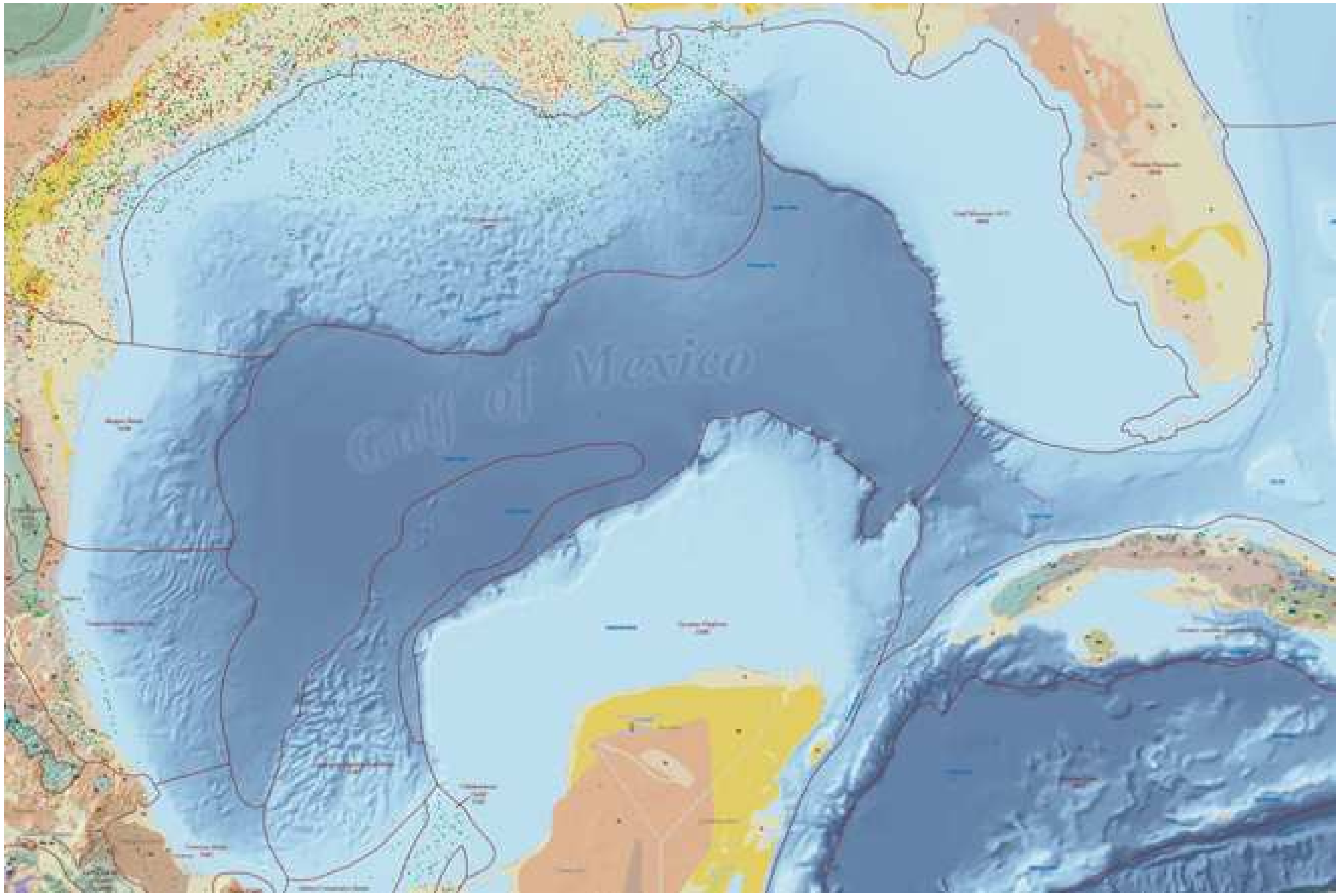
Scheme of scales



Why it is important?



Purpose of wave forecasting



Kinetic equation

The pair correlation function for excitations N_k obeys the kinetic equation (Nordheim,1929; Hasselmann, 1962; Zakharov, 1966)

$$\frac{\partial N_k}{\partial t} = st(N,N,N) + f_p(k) - f_d(k), \tag{1}$$

Here

$$\begin{aligned} st(N,N,N) = 4\pi \int \left| T_{\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3} \right|^2 \times \\ \times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - \\ - N_k N_{k_1} N_{k_3}) \delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{k}_3. \end{aligned} \tag{2}$$

The kinetic equation and its modifications are the base for all wave forecasting models.

White capping.

Dissipation function.

Dissipative part of kinetic equation

$$\frac{\partial N_{\vec{k}}}{\partial t} = ... + \gamma^{kin}_{\vec{k},\mu} \omega_k N_{\vec{k}}. \tag{3}$$

If $N_{\vec{k}}$ is almost monochromatic (swell) we can find dependence of γ^{kin} on average steepness μ:

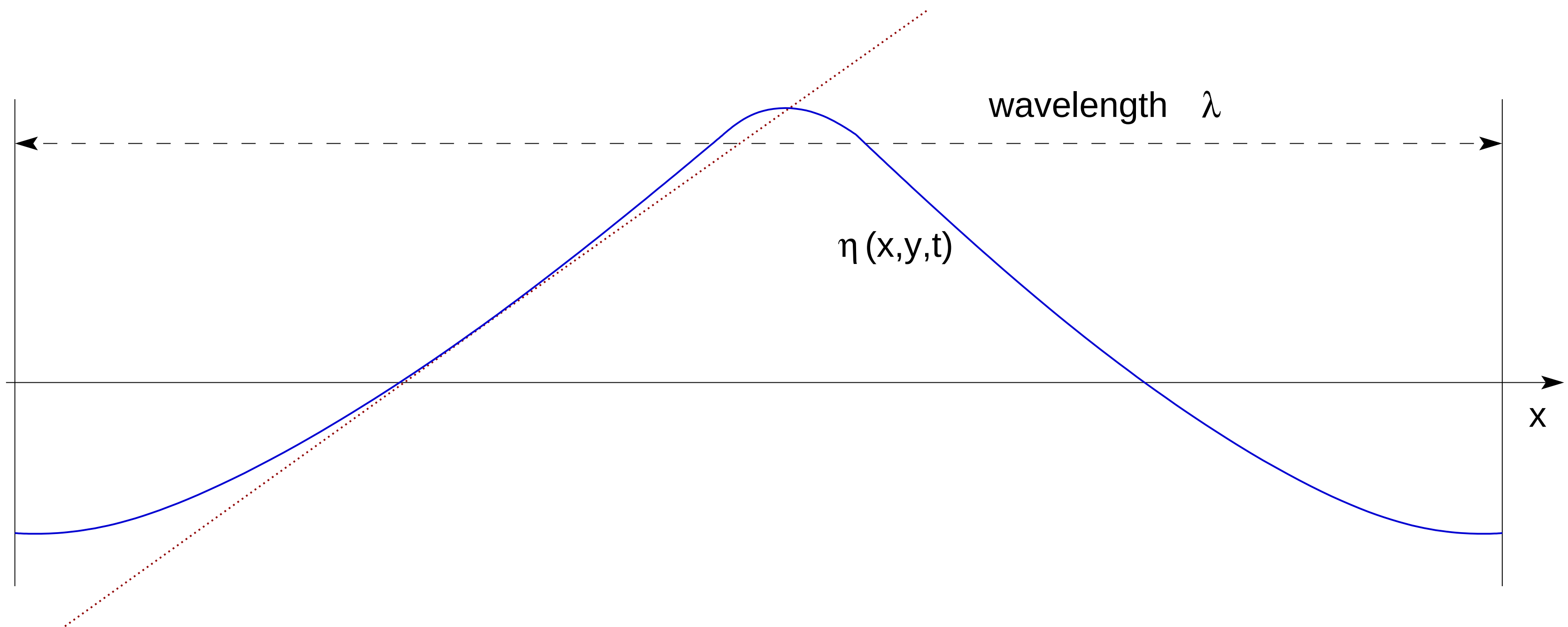
$$\gamma^{kin}(\mu) = \frac{\dot{N}}{\omega_p N}, \tag{4}$$

Here

$$N = \int n_{\vec{k}} \mathrm{d}^2 k.$$

Problem formulation

Let us consider a potential flow of an ideal fluid of infinite depth with a free surface. We use standard notations for velocity potential $\phi(\vec{r}, z, t), \vec{r} = (x, y); \vec{v} = \nabla\phi$ and surface elevation $\eta(\vec{r}, t)$.



Steepness of the surface $\mu = \sqrt{\langle|\nabla\eta(\vec{r}, t)|^2\rangle}$ — average slope of the surface.

Energy of the system

Fluid flow is incompressible $(\nabla \vec{v}) = \Delta\phi = 0$. The total energy of the system can be presented in the following form

$$H = T + U,$$

Kinetic energy:

$$T = \frac{1}{2}\int \mathrm{d}^2 r \int\limits_{-\infty}^{\eta} (\nabla\phi)^2 \mathrm{d}z, \tag{5}$$

Potential energy due to gravity:

$$U = \frac{1}{2} g \int \eta^2 \mathrm{d}^2 r, \tag{6}$$

here g is the gravity acceleration.

Hamiltonian expansion

It was shown by Zakharov (1966) that under these assumptions the fluid is a Hamiltonian system

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \tag{7}$$

where $\psi = \phi(\vec{r}, \eta(\vec{r}, t), t)$ is a velocity potential on the surface of the fluid. In order to calculate the value of ψ we have to solve the Laplace equation in the domain with varying surface η. One can simplify the situation, using the expansion of the Hamiltonian in powers of "steepness" (here $\Delta = \nabla^2$ and $\hat{k} = \sqrt{-\Delta}$)

$$\begin{aligned} H = & \frac{1}{2} \int \left(g\eta^2 + \psi \hat{k} \psi \right) \mathrm{d}^2 r + \\ & + \frac{1}{2} \int \eta \left[|\nabla \psi|^2 - (\hat{k}\psi)^2 \right] \mathrm{d}^2 r + \\ & + \frac{1}{2} \int \eta (\hat{k}\psi) \left[\hat{k}(\eta(\hat{k}\psi)) + \eta \Delta \psi \right] \mathrm{d}^2 r. \end{aligned} \tag{8}$$

Dynamical equations

In this case dynamical equations acquire the following form

$$
\begin{aligned}
\dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\
&\quad + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \tfrac{1}{2}\Delta[\eta^2\hat{k}\psi] + \tfrac{1}{2}\hat{k}[\eta^2\Delta\psi] - D_{\vec{r}}, \\
\dot{\psi} &= -g\eta - \tfrac{1}{2}\left[(\nabla\psi)^2 - (\hat{k}\psi)^2\right] - \\
&\quad -[\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi - D_{\vec{r}} + F_{\vec{r}}.
\end{aligned} \tag{9}
$$

Here $D_{\vec{r}}$ is some artificial damping term used to provide dissipation at small scales; $F_{\vec{r}}$ is a pumping term corresponding to external force (having in mind wind blow, for example). Let us introduce Fourier transform

$$
\psi_{\vec{k}} = \frac{1}{2\pi}\int \psi_{\vec{r}} \mathrm{e}^{\mathrm{i}\vec{k}\vec{r}} \mathrm{d}^2 r, \quad \eta_{\vec{k}} = \frac{1}{2\pi}\int \eta_{\vec{r}} \mathrm{e}^{\mathrm{i}\vec{k}\vec{r}} \mathrm{d}^2 r.
$$

Numerical scheme parameters

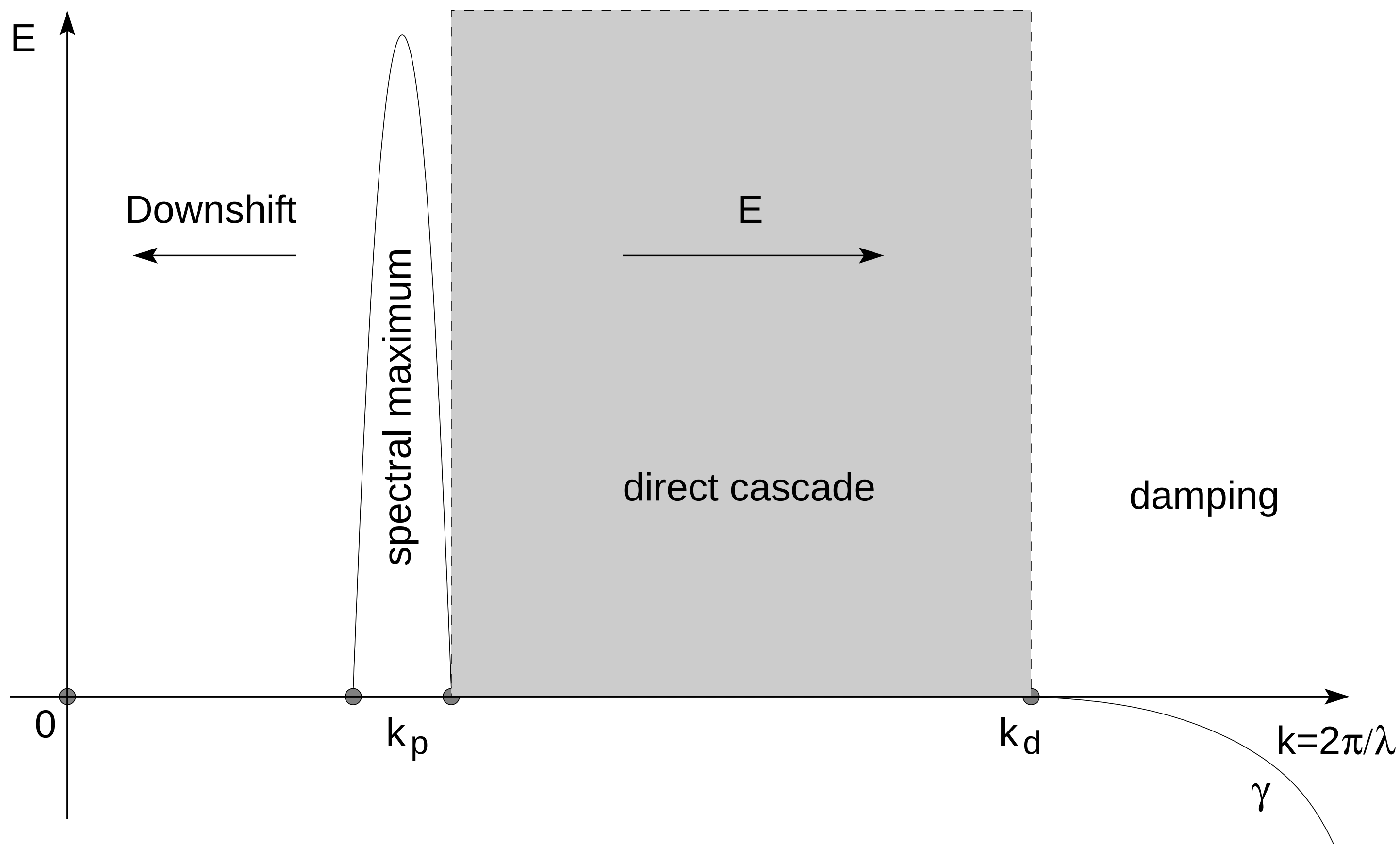
Let us add pseudo-viscous damping in dynamical equations

$$\begin{aligned}\dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\ &+ \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \frac{1}{2}\Delta[\eta^2\hat{k}\psi] + \frac{1}{2}\hat{k}[\eta^2\Delta\psi] - F^{-1}[\gamma_k\eta_{\vec{k}}], \\ \dot{\psi} &= -g\eta - \frac{1}{2}\left[(\nabla\psi)^2 - (\hat{k}\psi)^2\right] - \\ &- [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi - F^{-1}[\gamma_k\psi_{\vec{k}}].\end{aligned} \quad (10)$$

$$\begin{aligned}&\gamma_k = \gamma_0(k-k_d)^2, k > k_d, \gamma_0 = 2.86\times 10^{-3}; \\ &\gamma_k = 0, k \le k_d.\end{aligned} \quad (11)$$

Gravity acceleration $g = 1$. Simulation region $L_x = L_y = 2\pi$ with double periodic boundary conditions. Rectangular numerical grid $N_x = 512$, $N_y = 4096$. Pseudo-viscous dissipation starts at $k_d = 1024$. Time step $\Delta t = 4.23\times 10^{-4}$

Scheme of scales

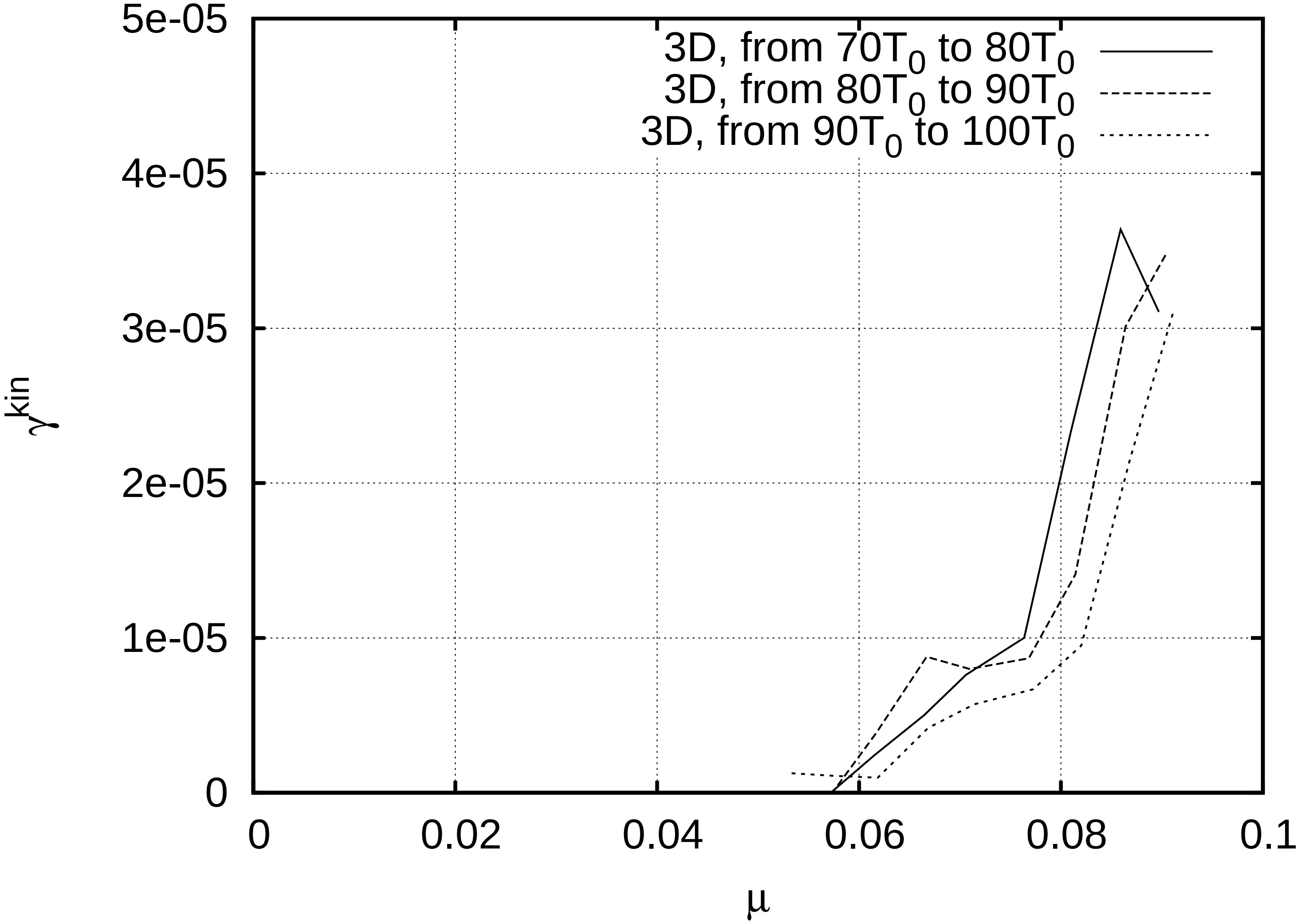


Initial conditions.

Gauss-shaped spectrum, centered at $\vec{k} = (0; 100)$ with width $D = 30$.

$$\begin{cases} |a_{\vec{k}}| = A_i \exp\left(-\frac{1}{2}\frac{\left|\vec{k}-\vec{k}_0\right|^2}{D_i^2}\right), \left|\vec{k}-\vec{k}_0\right| \le 2D_i, \\ |a_{\vec{k}}| = 10^{-12}, \left|\vec{k}-\vec{k}_0\right| > 2D_i. \end{cases} \tag{12}$$

Dissipation function. The first experiment.



Proposed energy transfer mechanism. The first experiment.

Mechanism:

- High steepness $\rightarrow$ wide spectrum $\rightarrow$ energy quickly delivered to the dissipation region and dissipated completely.

Problem:

- Weakly nonlinear model $\rightarrow$ we cannot model wavebreaking or whitecapping in details which are strongly nonlinear phenomena.

Model of energy transfer mechanism. The first experiment.

Remedy for a problem:

- We don't need to know wavebreaking **details**, because we need to know how much energy was dissipated, instead of how in details it was dissipated.
- Multiple harmonics generation describes spectrum widening during early stage of wavebreaking and whitecapping. This nonresonant mechanism is taken into account in our dynamic equations.
- Due to the universal mechanism of the spectrum widening we can check our results in the fully nonlinear 2D-model, result should be the same.

Fully nonlinear 2D experiment.

Suppose that incompressible fluid covers the domain

$$-\infty < y < \eta(x,t).$$

The flow is potential and incompressible, hence $v = \nabla\phi, \nabla v = 0, \Delta\phi = 0.$

$$H = T + U,$$

Kinetic energy:

$$T = \frac{1}{2}\int \mathrm{d}x \int\limits_{-\infty}^{\eta} (\nabla\phi)^2 \mathrm{d}y, \qquad (13)$$

Potential energy due to gravity:

$$U = \frac{1}{2}g\int \eta^2 \mathrm{d}x \qquad (14)$$

Hamiltonian equations.

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \tag{15}$$

One can perform the conformal transformation to map the domain that is filled with fluid,

$$-\infty < x < +\infty,\ -\infty < y < \eta(x,t),\ z = x + \mathrm{i}y,$$

in z-plane to the lower half-plane

$$-\infty < u < +\infty,\ -\infty < v < 0,\ w = u + \mathrm{i}v,$$

in w-plane.

Hilbert transformation.

Now, the shape of surface $\eta(x,t)$ is presented by parametric equations

$$y = y(u,t),\ x = x(u,t).$$

where $x(u,t)$ and $y(u,t)$ are related through Hilbert transformation

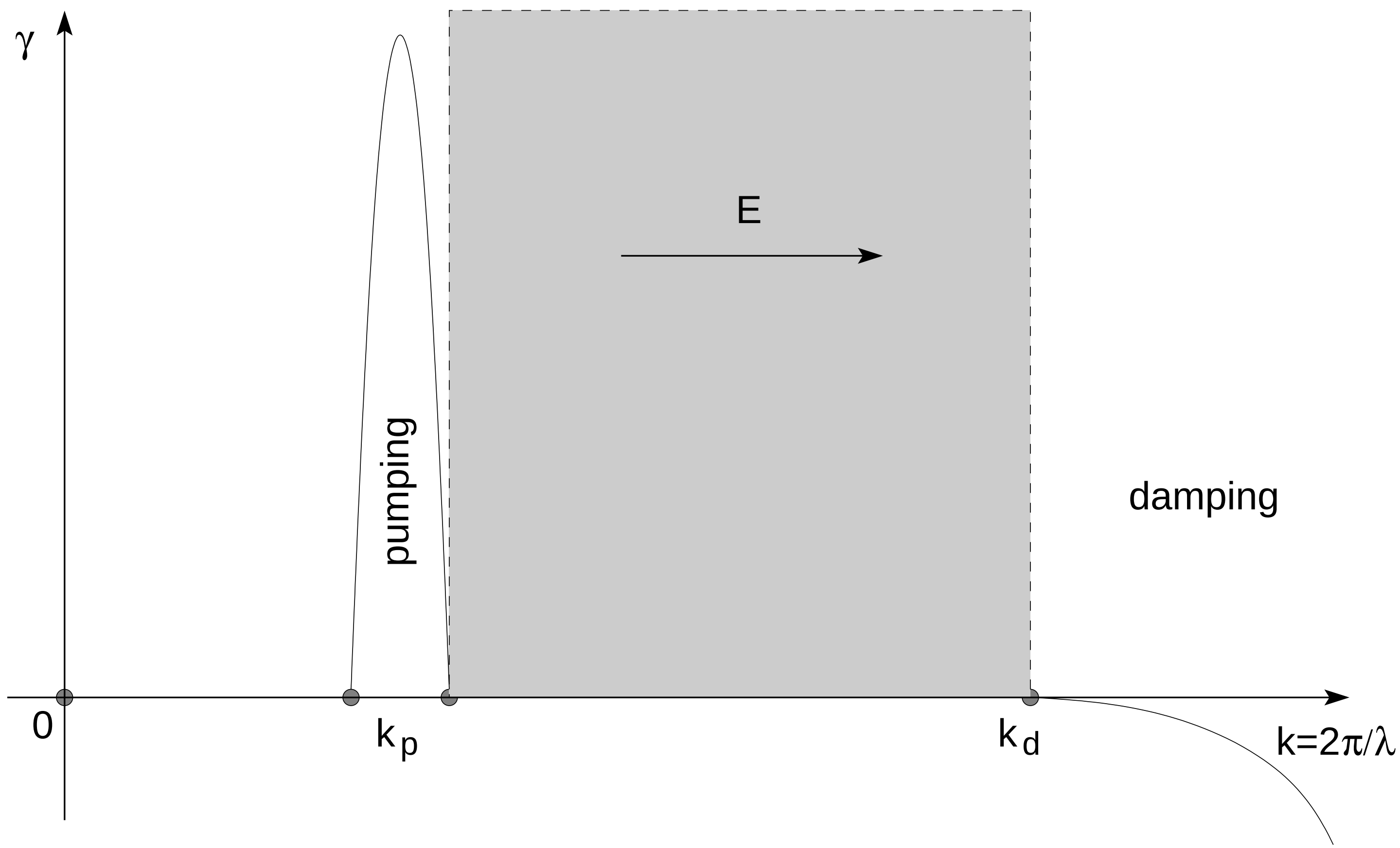
$$y(u,t) = \widehat{H}(x(u,t) - u),\ x(u,t) = u - \widehat{H}(y(u,t)).$$

$$\widehat{H}(f(u)) = \frac{1}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{f(u')\mathrm{d}u}{u' - u}$$

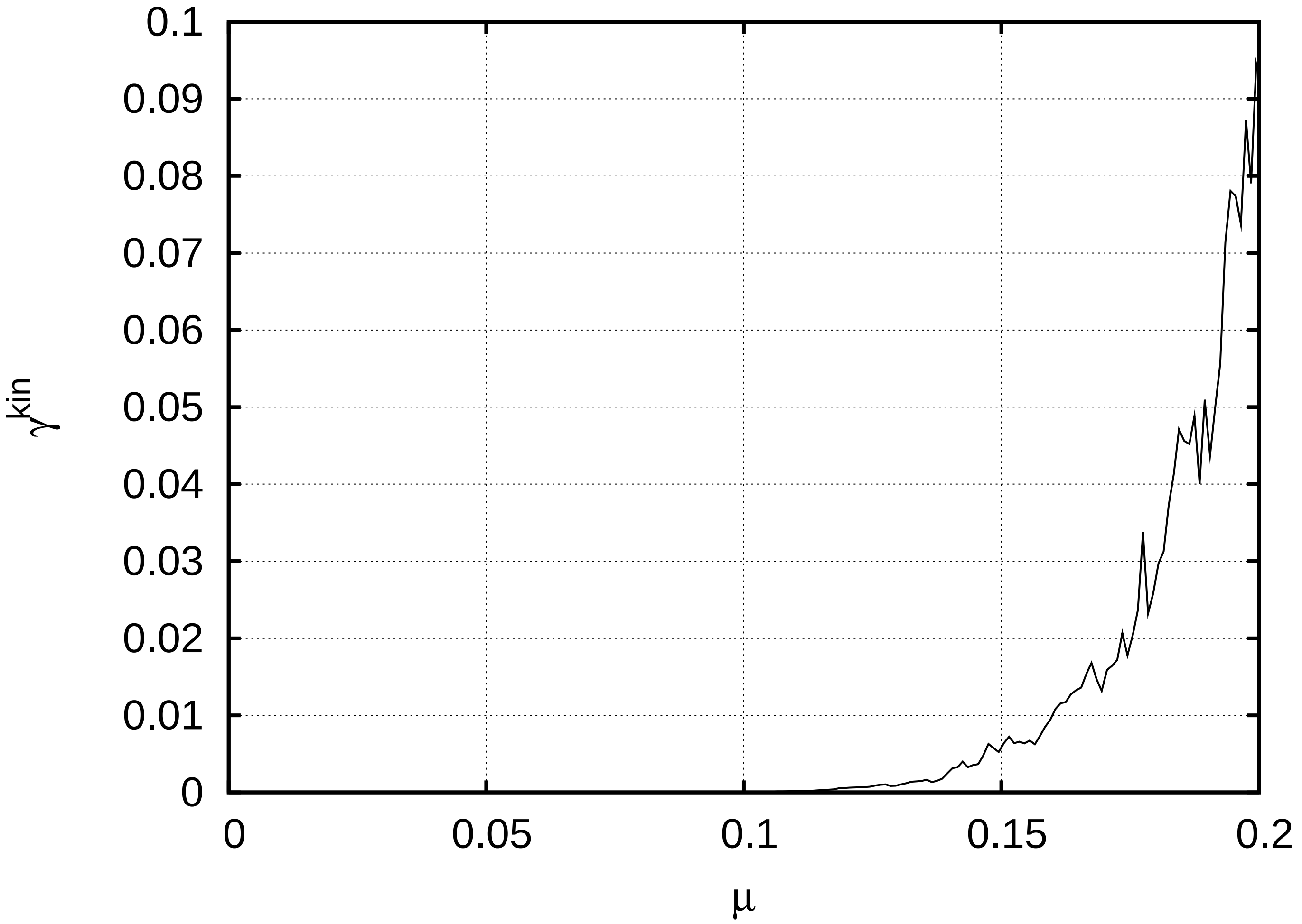
Kinetic energy term in new variables:

$$\left.\frac{\partial \Phi}{\partial v}\right|_{v=0} = -\frac{\partial}{\partial u}\widehat{H}\Phi.$$

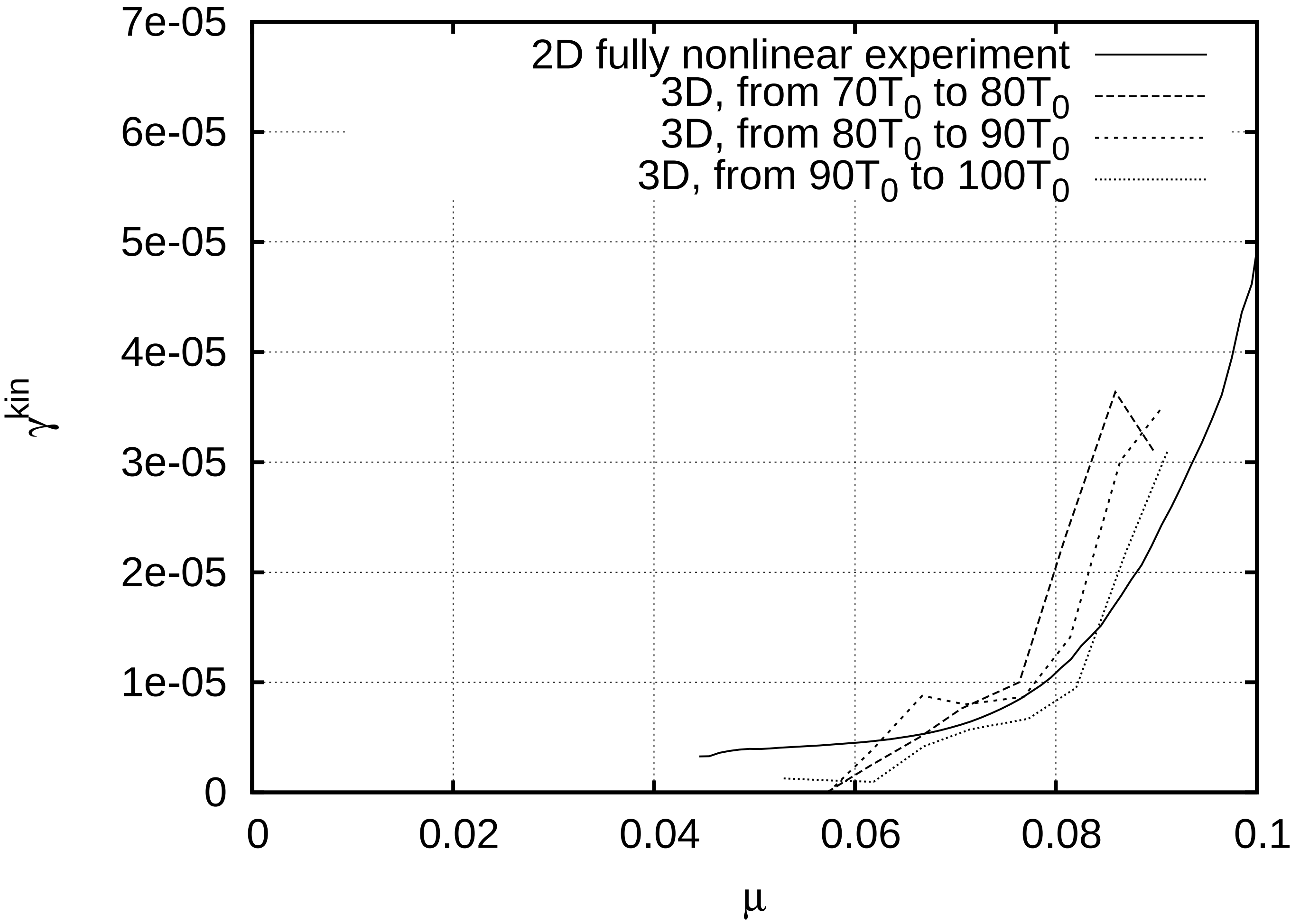
Scheme of scales



Dissipation function. The second experiment.



Dissipation function. Both experiments.



Waves forecasting models.

$$\gamma_{\vec{k}} = C_{ds}\tilde{\omega}\frac{k}{\tilde{k}}\left((1-\delta)+\delta\frac{k}{\tilde{k}}\right)\left(\frac{\tilde{S}}{\tilde{S}_{pm}}\right)^{p} \quad (16)$$

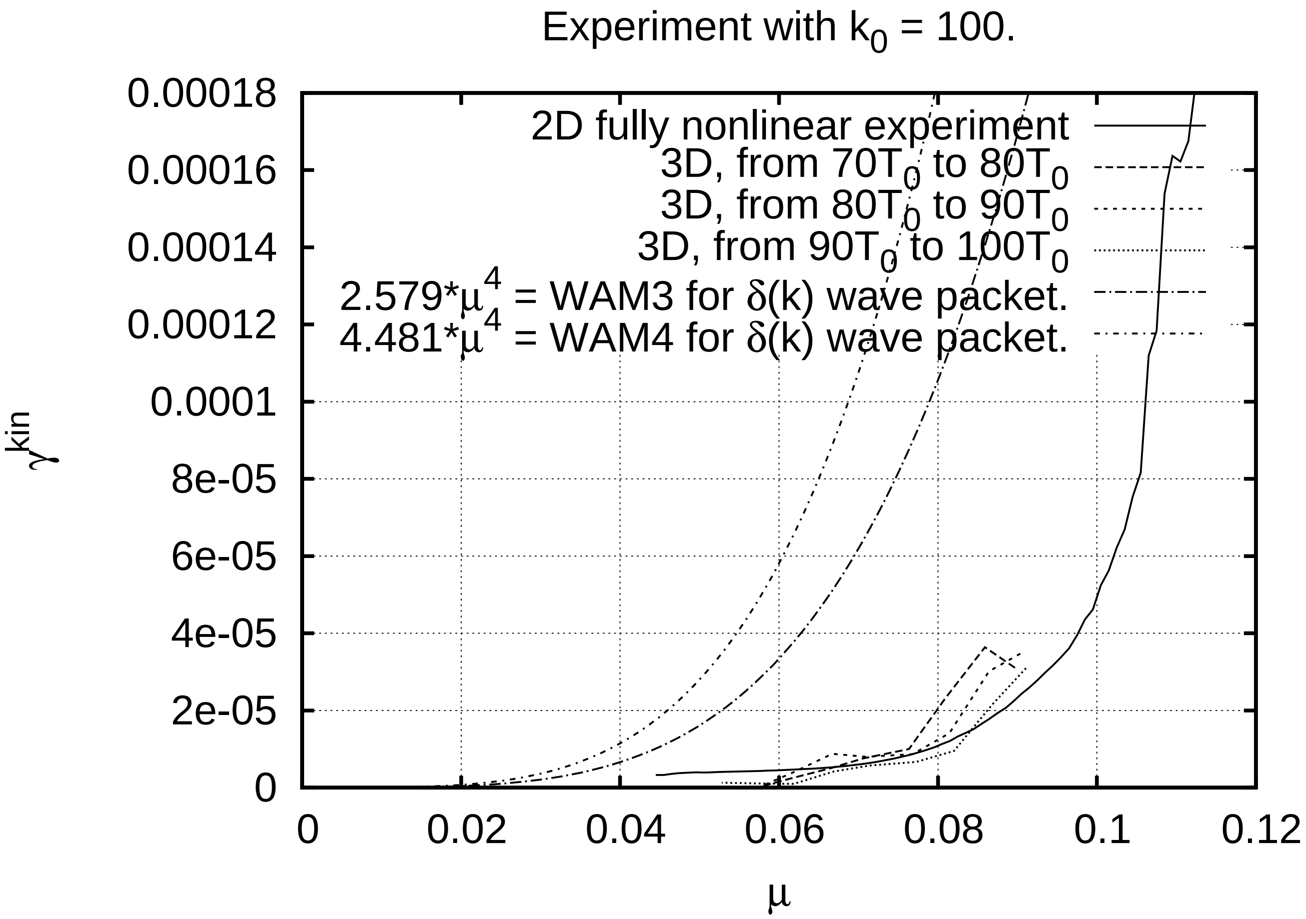
where k and ω are the wave number and frequency, tilde denotes mean value; C_{ds}, δ and p are tunable coefficients; $S = \tilde{k}\sqrt{H}$ is the overall steepness; $\tilde{S}_{PM} = (3.02\times10^{-3})^{1/2}$ is the value of $\tilde{S}$ for the Pierson-Moscowitz spectrum (note that the characteristic steepness is $\mu = \sqrt{2}S$). The values of the tunable coefficients for the *WAM3* case are:

$$C_{ds} = 2.35 \times 10^{-5}, \;\; \delta = 0, \;\; p = 4 \quad (17)$$

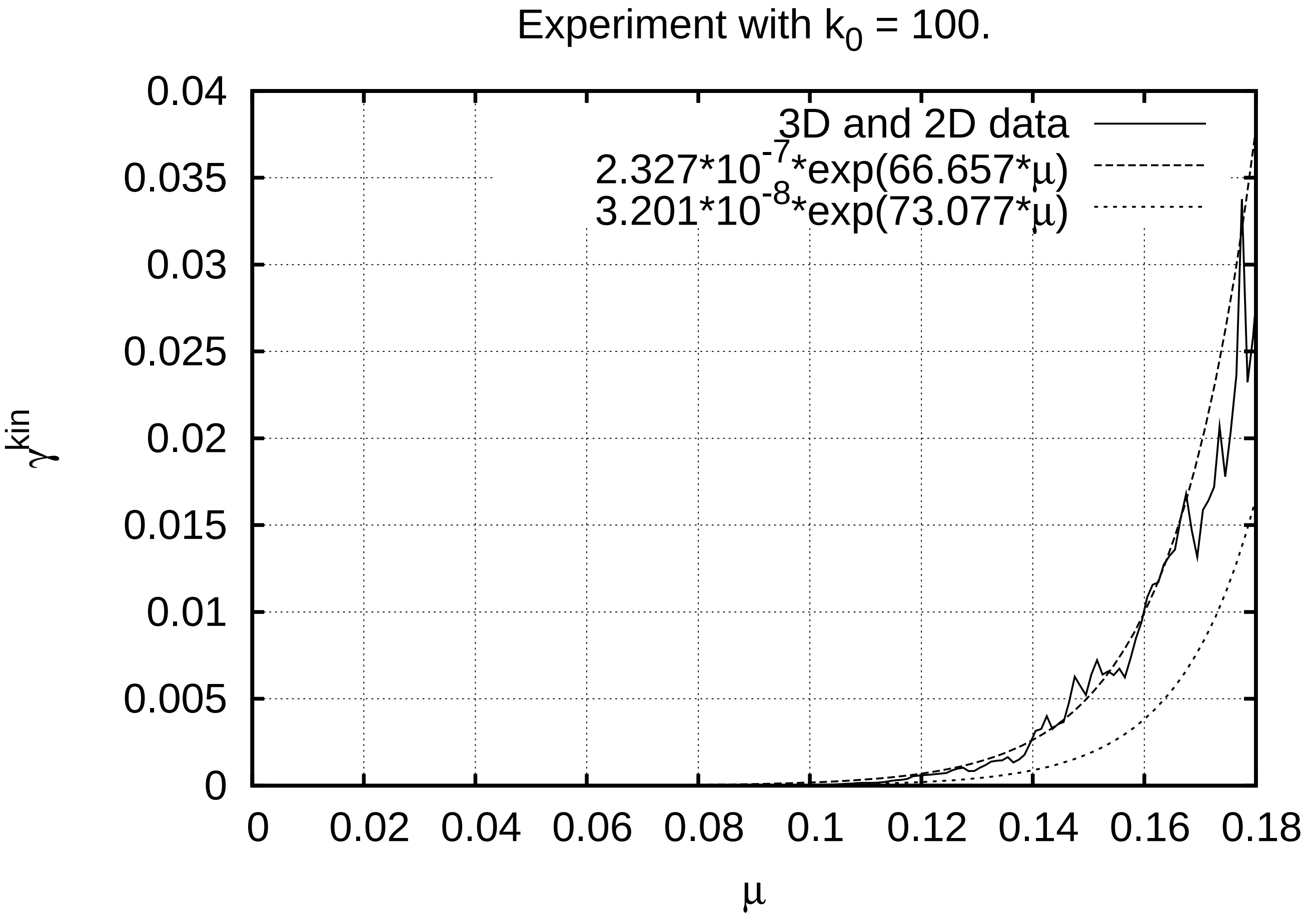
and for the *WAM4* case are:

$$C_{ds} = 4.09 \times 10^{-5}, \;\; \delta = 0.5, \;\; p = 4 \quad (18)$$

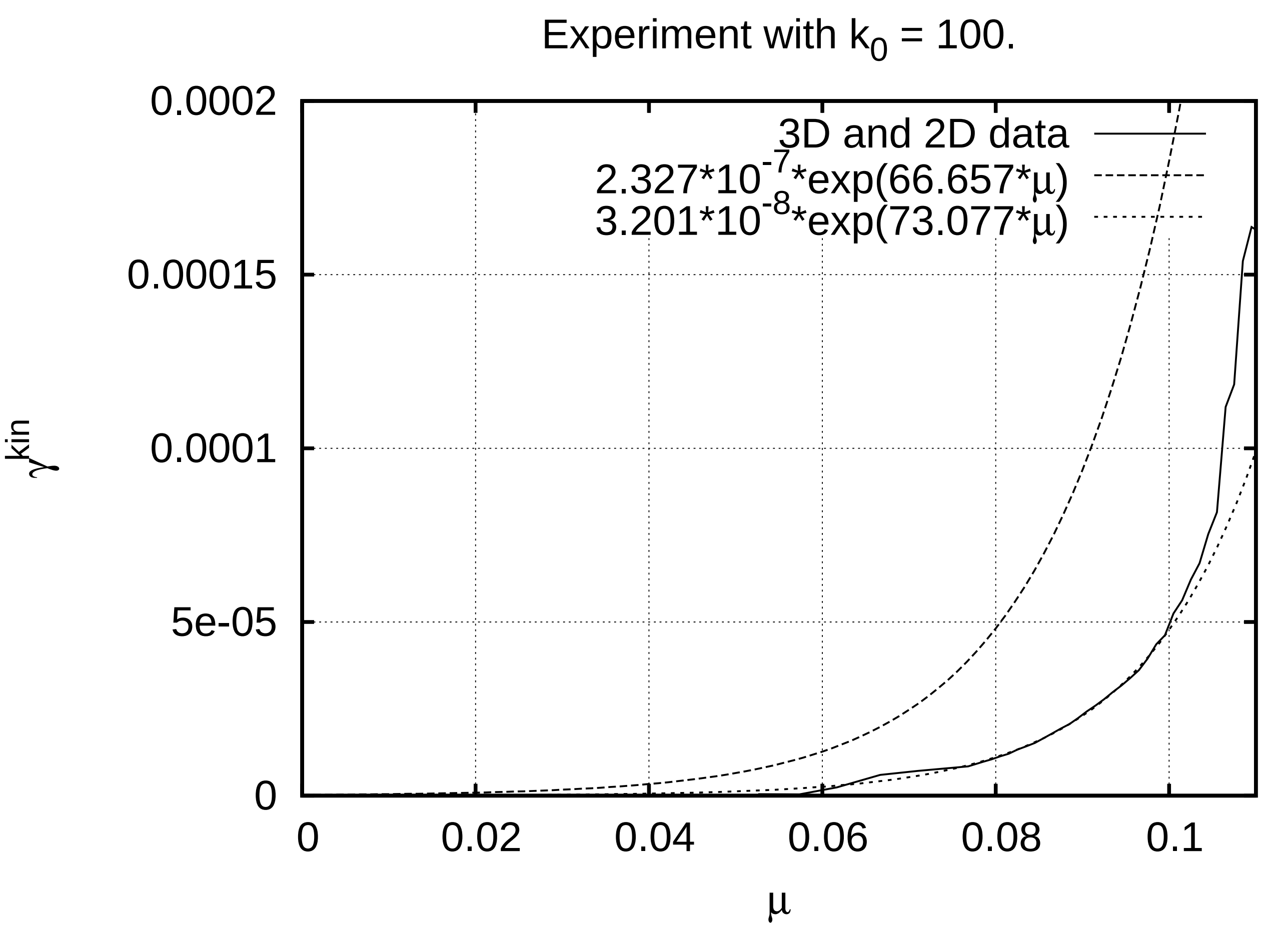
Dissipation function. Comparison with waves forecasting models.



Dissipation function. Exponential fit.



Dissipation function. Exponential fit. Low μ.



Results.

- Performed simulation of the gravity waves decaying turbulence in 2D fully-nonlinear and 3D weakly-nonlinear models.
- Obtained dependence of the dissipation function on average steepness.
- Demonstrated threshold-like character of the dissipation due to whitecapping.
- Results are significantly different with respect to wave-forecasting models terms.