Kirill Zybin

Lagrangian and Eulerian statistical theory of hydrodynamic turbulence

P.N. Lebedev Physical Institute of RAS

Eulerian and Lagrangian structure functions

- Eulerian transversal structure functions:

$$S_n^{\perp}(l) = \left\langle \left| (\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})) \times \frac{\mathbf{l}}{l} \right|^n \right\rangle \propto l^{\zeta_n^E}$$

- Lagrangian structure functions:

$$S_n^L(\tau) = \langle |\mathbf{v}(t+\tau) - \mathbf{v}(t)|^n \rangle \propto \tau^{\zeta_n^L}$$

- Modern experiment and numerical calculations
  $S_n :\ n \sim 8 - 10$
- there is **no** theory based on Navier Stokes equation

Kolmogorov (K41) theory

- stationary, locally isotropic and homogeneous turbulence in incompressible fluid
- inertial range – dimensional theory

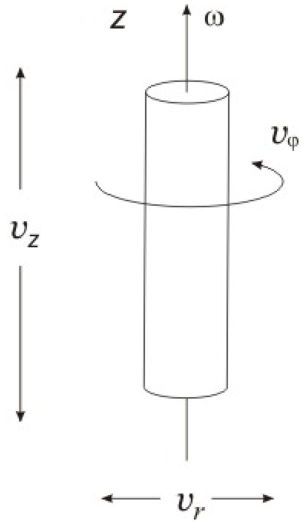
$\eta \ll l \ll L$– Eulerian case $\qquad$ $\tau_\eta \ll \tau \ll T$– Lagrangian case

- structure functions

$\zeta_n^E = n/3$– Eulerian case $\qquad$ $\zeta_n^L = n/2\,,$– Lagrangian case

experiment – anomalous scaling

Uniform cylindrical flow



Cylindrical flow

$$v_\phi = \omega r\,, \qquad v_r = ar\,, \qquad v_z = bz\,, \qquad p = \frac{1}{2}P_1(t)r^2 + \frac{1}{2}P_2(t)z^2$$

after introducing into Navier-Stokes equation

$$2a + b = 0\,, \qquad \dot{a} + a^2 - \omega^2 = -P_1$$

$$\dot{\omega} + 2a\omega = 0\,, \qquad \dot{b} + b^2 = -P_2$$

We have 5 functions and only 4 equations. Across the cylinder we have an algebraic equation – pressure balance
The set of equations is reduced to

$$\ddot{\omega} = -P_2(t)\omega$$

$P_2(t)$ is rather complicated "random" function and its time average is zero, the time intervals when $P_2(t) > 0$ and $P_2(t) < 0$ are equally probable. However, at $P_2(t) > 0$ the function ω oscillates, the oscillation amplitude changes weakly. On the contrary, at $P_2(t) < 0$,

the function ω grows exponentially.

Limitation of uniform approximation

$$\dot{R} = a\,R\,, \qquad \dot{\omega} = -2a\,\omega$$

Here $R(t)$ is the radius of the cylinder

$$\frac{\dot{R}}{R} \propto -\frac{\dot{\omega}}{2\omega} \quad \rightarrow \quad R(t) \propto \omega^{-1/2}$$

The limitation of linear expansion is defined by relation

$$\omega(t) R_l \leq U_0$$

Here U_0 is the large scale pulsation velocity. We see that $R_l(t) \propto 1/\omega$. Let us consider relation between boundary of the vortex and the boundary of linear expansion

$$\frac{R_l}{R} \propto \frac{1}{\sqrt{\omega}} \rightarrow 0\,, \quad \text{if} \quad \omega \rightarrow \infty$$

Thus, the uniform approximation improves in time

Strong vortex filament $r_0 << L$

$$V_i = \epsilon_{iml} r_m q_l \phi(r_\perp)$$

Here

$$q_l = \omega_l/\omega \qquad\qquad r_\perp = \sqrt{r^2 - (q_i r_i)^2}$$

and function $\phi(r_\perp)$ is defined by local vorticity

$$\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}\left(r_\perp^2 \phi(r_\perp)\right) = \omega(r_\perp)$$

The dynamics of vorticity is defined by symmetric tensor

$$\dot{\omega}_i = b_{ik}\omega_k \,, \qquad b_{ik} = \frac{1}{2}\left(\frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i}\right)$$

for strong vortex filament b_{ik} is equal:

$$b_{ik} = \left[\epsilon_{iml} r_m q_l \left(\frac{r_k}{r_\perp} - q_k \frac{r_n q_n}{r_\perp}\right) + \epsilon_{kml} r_m q_l \left(\frac{r_i}{r_\perp} - q_i \frac{r_n q_n}{r_\perp}\right)\right] \frac{d\phi}{dr_\perp}$$

$b_{ik}\omega_k = 0$, the changes of ω is due to large scale pulsations only

Statistical description

- In general case evolution of vorticity along a particle trajectory is described by the equation

$$\ddot{\omega}_n = -\rho_{nk}(t)\omega_k \tag{1}$$

  Here $\rho_{nk} = \nabla_n \nabla_k p$, and p is the pressure
- only the large-scale component of ρ_{nk} must be taken into account
- Hence, ρ_{nk} is independent on the local value of ω in vortex filaments, and the equation (1) becomes linear
- Let $\rho_{nk}(t)$ be a stochastic process

Equation for PDF

Liouville equation $\tilde{f}(\omega_i, \nu_i, t)$:

$$\frac{\partial \tilde{f}}{\partial t} + \nu_i \frac{\partial \tilde{f}}{\partial \omega_i} = \rho_{ik} \omega_i \frac{\partial \tilde{f}}{\partial \nu_k}$$

here $\nu_i = \dot{\omega}_i$
Introducing the average part $f = < \tilde{f} >$ and the fluctuations δf :

$$\tilde{f} = f + \delta f$$

iterating on $|\delta f|$ one can find equation for $f(\omega, \nu, t)$ as an infinite series:

$$\frac{\partial f}{\partial t} + \nu_i \frac{\partial f}{\partial \omega_i} = L_1(f) + L_2(f) + ...$$

$$L_1(f) =$$

$$\omega_i \int_{-\infty}^{t} \left\langle \rho_{ik}(t)\rho_{lm}(t') \right\rangle \delta(\omega - \omega' - \nu(t-t'))\omega'_l \frac{\partial^2}{\partial \nu_k \partial \nu_m} f(\omega', \nu, t')\, dt'\, d\omega'$$

The term $L_2(F)$ sketchy looks like:

$$L_2(f) = \omega \frac{\partial}{\partial \nu} \int_0^\infty \delta(\omega - \omega_1 - \nu\tau_1) d\tau_1$$

$$\omega_1 \frac{\partial}{\partial \nu} \int_0^\infty D(\tau_1)\delta(\omega_1 - \omega_2 - \nu\tau_2) d\tau_2$$

$$\omega_2 \frac{\partial}{\partial \nu} \int_0^\infty D(\tau_3)\delta(\omega_2 - \omega_3 - \nu\tau_3)$$

$$\omega_3 \frac{\partial}{\partial \nu} f(\omega_3, \nu, t - \tau_1 - \tau_2 - \tau_3)\, d\tau_3\, d\omega_1 d\omega_2 d\omega_3$$

The parameter $D(\tau) =< \rho(t+\tau)\rho(t) >$.

Some properties of PDF

- The moments
  $M_P(n,m,k) = \left\langle \omega^{2n}\nu^{2m}(\omega\nu)^k \right\rangle \equiv \int \omega^{2n}\nu^{2m}(\omega\nu)^k f d\omega d\nu\,,$
  we obtain a closed set of linear differential equations
- statistical moments grow exponentially $\langle|\omega|^n\rangle \propto e^{\Lambda_n t}$
- Let τ_g be the correlation time of the function $D(\tau)$
- if $\Lambda_n\tau_g \gg 1$ it is true for $n\to\infty$ $\Lambda_n = \lambda n\,, \qquad \lambda = \tau_g^{-1/4}$
- if $\Lambda_n\tau_g \ll 1$

$$\begin{array}{lll} \Lambda_2 = 2.52\,, & \Lambda_4 = 6.12\,, & \Lambda_6 = 10.43\,, \\ \Lambda_8 = 15.25\,, & \Lambda_{10} = 20.48\,, & \Lambda_{12} = 26\,, \\ \Lambda_{14} = 32.03\,, & \Lambda_{16} = 38.25\,, & \Lambda_{18} = 44.73\,, \\ \Lambda_{20} = 51.46\,, & \Lambda_{22} = 58.42\,, & \Lambda_{24} = 65.58\,, \end{array}$$

$$\Lambda_{26} = 72.95\,,\ \Lambda_{28} = 80.52\,,\ \Lambda_{30} = 88.26\,,\ \Lambda_{32} = 96.16$$

$\Lambda_{2n} > n\Lambda_2$ — **intermittency**

Transversal Eulerian structure functions $S_n^{\perp}(l)$

Transversal Eulerian correlation functions have the form:

$$S_n^{\perp}(l) = \langle |\delta \mathbf{v}_{\perp}(l)|^n \rangle$$

inside the vortex it could be presented in the form:

$$S_n^{\perp}(l) = \langle |\mathbf{l}|^n |\omega|^n \rangle = l^n \langle |\omega|^n \rangle \propto l^n e^{\Lambda_n t}$$

all these correlations are growing with time exponentially, but we have some limitation

$$\langle |\delta \mathbf{v}|_{\perp}^{\mathbf{n}} \rangle = l^n e^{\Lambda_n t} \leq U^n << (\delta U_{max})^n$$

One can see that the smallest time is

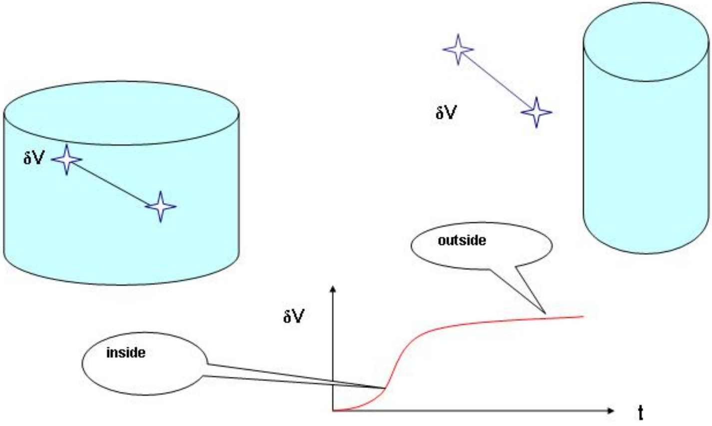
$$t^*(l) = 1/\lambda \, ln\,(U/l)$$

After this time the correlation practically does not change

$$S_n^{\perp}(l) =\propto l^{\zeta_n(E)}, \qquad \zeta_n(E) = n - \frac{\Lambda_n}{\lambda}$$

One can see that cutting parameter U determines the amplitude of structure function but not the scaling law

Saturation of correlation function. Qualitative model.



Lagrangian structure functions $S_n(\tau)$

- Eulerian structure functions were calculated for given difference **l** between two points.
- In Lagrangian case you should repeat the same procedure for given time difference along trajectory τ
- One can see that in the vortex lagrangian particles oscillate and velocity difference could be presented in the form:

$$\delta v_{\perp} = \omega(t)\, r_0 \; Sin\left(\int_t^{t+\tau} \omega(t_1)\, dt_1\right), \qquad r_0 \approx const$$

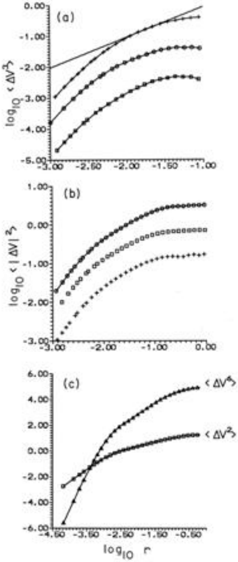
Using these relations we can rewrite for small τ in the form:

$$S_n(\tau, L) \propto \left\langle \tau^n \omega^{2n} \right\rangle = \tau^n \left\langle \omega^{2n} \right\rangle$$

- repeating the previous procedure one can find:

$$S_n(\tau) \propto \tau^{\zeta_n(L)}, \qquad \zeta_n(L) = n - \frac{\Lambda_{2n}}{2\lambda}$$

Euler Extended Self-Similarity Ansatz (Benzi 1993)



Cylinder (a), (b)

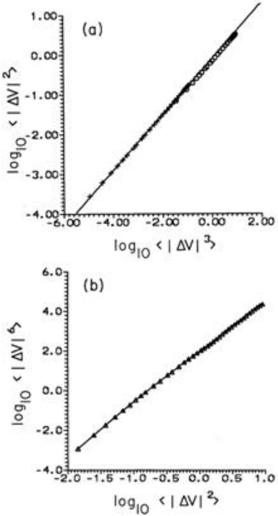
Re = 6000 –

Re = 22500 –

Re = 47000 –

Jet (c)

Re = 300000



Comparison between the theory, numerical simulation and experiment

- In modern numerical simulations and experiments the scaling parameters are determined using "Self-Similarity Ansatz"
- The theory has only one unknown parameter λ or τ_g We determined this parameter, normalizing on $S_4(\tau)$ (Benzi et al arXiv:09050082)
- All the other scaling exponents are presented in the table

L-DNS		L-Theory		E-DNS	E-Theory	E-Exp
ζ_4/ζ_2	1.66	1.66	ζ_2/ζ_3	0.71 ± 0.01	0.72	0.7
ζ_6/ζ_2	2.10	2.14	ζ_4/ζ_3	1.26 ± 0.01	1.28	1.28
ζ_8/ζ_2	2.33	2.45	ζ_6/ζ_3	1.68 ± 0.03	1.74	1.77
ζ_{10}/ζ_2	2.45	2.64	ζ_8/ζ_3	1.98 ± 0.1	2.13	2.23
			ζ_{10}/ζ_3	2.25 ± 0.15	2.46	

Kolmogorov approach

Energy flux:

$$\varepsilon = \text{const}$$

Dimension theory

Our approach

Stochastic vortex instability

Vortex filaments

Kolmogorov approach		Our approach
Nonlinear cascade	← process →	Linear vortex stretching
Nonlinear cascade	← nonlinearity →	Balance across vortex filament

Turbulent structures are not determined neither by energy flux nor by dimensional consideration

$$\frac{\mathrm{dv}}{\mathrm{dt}} = -\frac{\nabla p}{\rho} + f \quad \longleftarrow \quad \text{Stochastic equation} \quad \longrightarrow \quad \frac{\mathrm{d}^2 \omega_i}{dt^2} = \rho_{ik}\omega_k$$

Conclusion

- Lagrangian and Eulerian velocity structure functions scaling exponents of high orders are derived in the inertial interval
- The comparison of the theory predictions with experimental and numerical results shows an excellent agreement
- We emphasize that only one fitting parameters was used for the determination of both set (Eulerian and Lagrangian) scaling exponents
- The power-law dependence of the Lagrangian structure functions was not suggested in the model
- All the obtained relations are the consequences of the equation for PDF derived directly from the Navier-Stokes equation

Alexander V. Gurevich

Valeria A. Sirota

Anton S. Ilyin