"Fast" evolution of wave turbulence

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Background

The fundamentals of the established understanding of evolution of random wave fields :

- Waves are weakly nonlinear, i.e. their characteristic steepness ε is small.
- Waves evolve primarily due to nonlinear interactions, rather than due to the direct effect of forcing.
- Due to the triad or quartet nonlinear interactions 'individual' wave amplitudes evolve respectively on the 'dynamic' $O(\varepsilon^{-1})$ or $O(\varepsilon^{-2})$ timescale.
- Random wind-wave fields are described by ensemble averaged quantities; then the $O(\varepsilon^{-1})$ or $O(\varepsilon^{-2})$ dynamics of the individual wave amplitudes averages out and the evolution of wave spectra occurs respectively on the 'kinetic' $O(\varepsilon^{-2})$ or $O(\varepsilon^{-4})$ timescale and is described by the kinetic equation.

- Further on, for brevity (and without much loss of generality) I will mainly speak about the case of non-decaying spectra taking wind waves as an exmple

  The kinetic equation prescribes the scaling of energy fluxes as $O(\varepsilon^6)$, and the existence of Kolmogorov-Zakharov (KZ) cascades of energy and wave action that gives rise to the development of powerlike wave spectra

This picture assumes that the wave fields are close to stationarity, and that there is spatial homogeneity of the environment. The picture is supported experimentally and by direct numerical simulations.

When these assumptions are violated (as often happens in nature), e.g. sharp change of wind, the standard kinetic theory cannot be applied and what happens is not known.

Plan

- What happens when the assumptions of stationarity, and spatial homogeneity of the environment are violated ?

  We show (by DNS and experimentally) that if a wave field is strongly perturbed then its adjustment occurs on the "fast" (dynamical) ε^{-2} timescale, rather than the kinetic ε^{-4} timescale.

- What happens when the assumptions of stationarity, and spatial homogeneity of the environment are NOT violated ?

  We show by DNS that the fast evolution still occurs.

- We generalize the KE to describe such scenarios, explain why and when fast evolution occurs and discuss some properties of the generalized KE (gKE).

DNS: Numerical algorithm

The algorithm was described in a number of papers by Annenkov & Shrira:

J.Fluid Mech., **449**, (2001), 341-371.

Phys. Review Letters, 96(20),(2006), 204501(4).

Phys. Review Letters, 102, (2009), 024502

Geophysical Research Letters, 36, L13603, (2009), doi:10.1029 2009GL038613.

Here we mention just the main points:

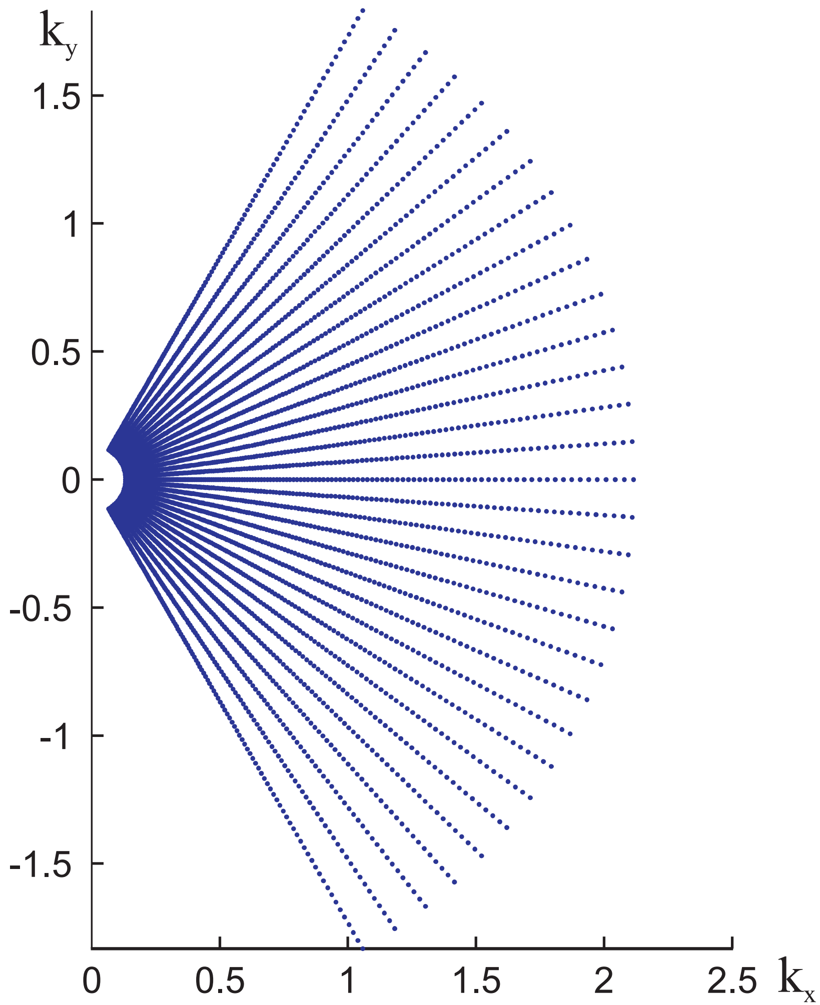
The algorithm is based upon direct integration of the "reduced" Zakharov equation

$$\mathrm{i}\frac{\partial b_0}{\partial t} \;=\; \omega_0 b_0 + \int T_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3}\, \mathrm{d}\mathbf{k}_{123} + \beta b_0$$

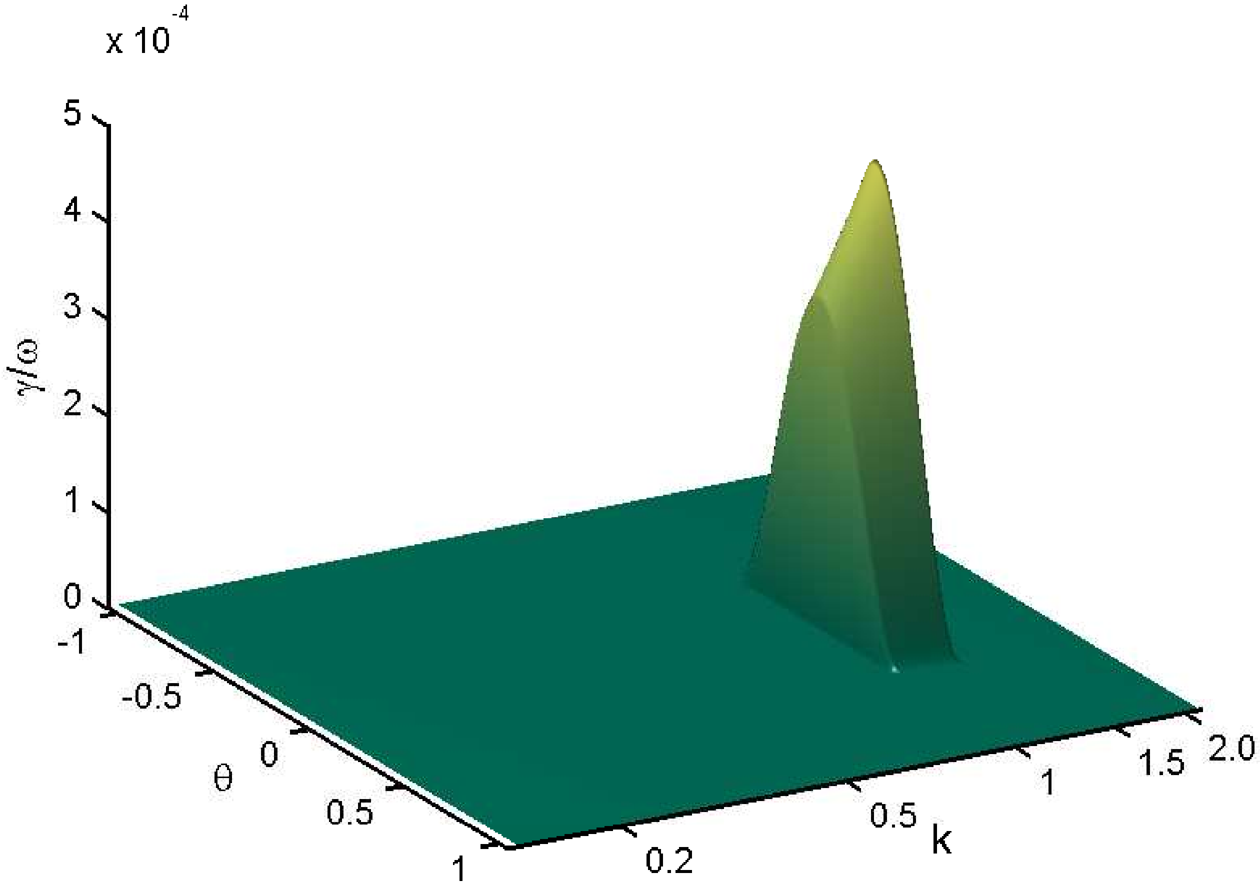
Notation: $\delta_{0+1-2-3} = \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, $\mathrm{d}\mathbf{k}_{123} = \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \mathrm{d}\mathbf{k}_3$

Direct numerical simulation

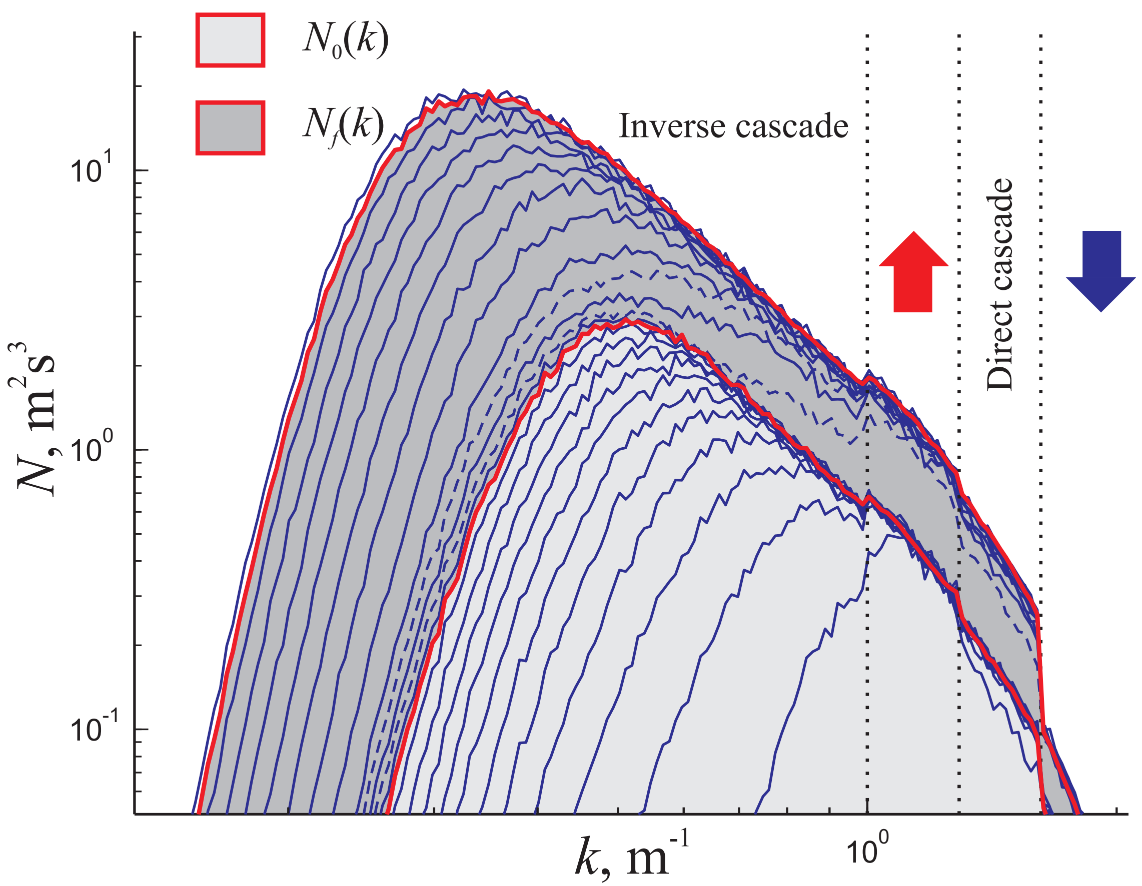
- performed within the framework of the reduced Zakharov equation
- grid consists of $O(10^4)$ *wave packets*
- initial phases are chosen randomly
- $O(10^8 - 10^9)$ exact and approximate resonant interactions
- logarithmic spacing in wavenumber (or frequency), so that the "density" of nonlinear interactions is homogeneous across the grid
- 4th-order Runge-Kutta algorithm is used
- evolution is computed for $O(10^4 - 10^5)$ periods
- averaging is over 30 realisations



Computational grid: 4991 wavepackets in the range $0.13 < k < 2.12$, angle 120^o.

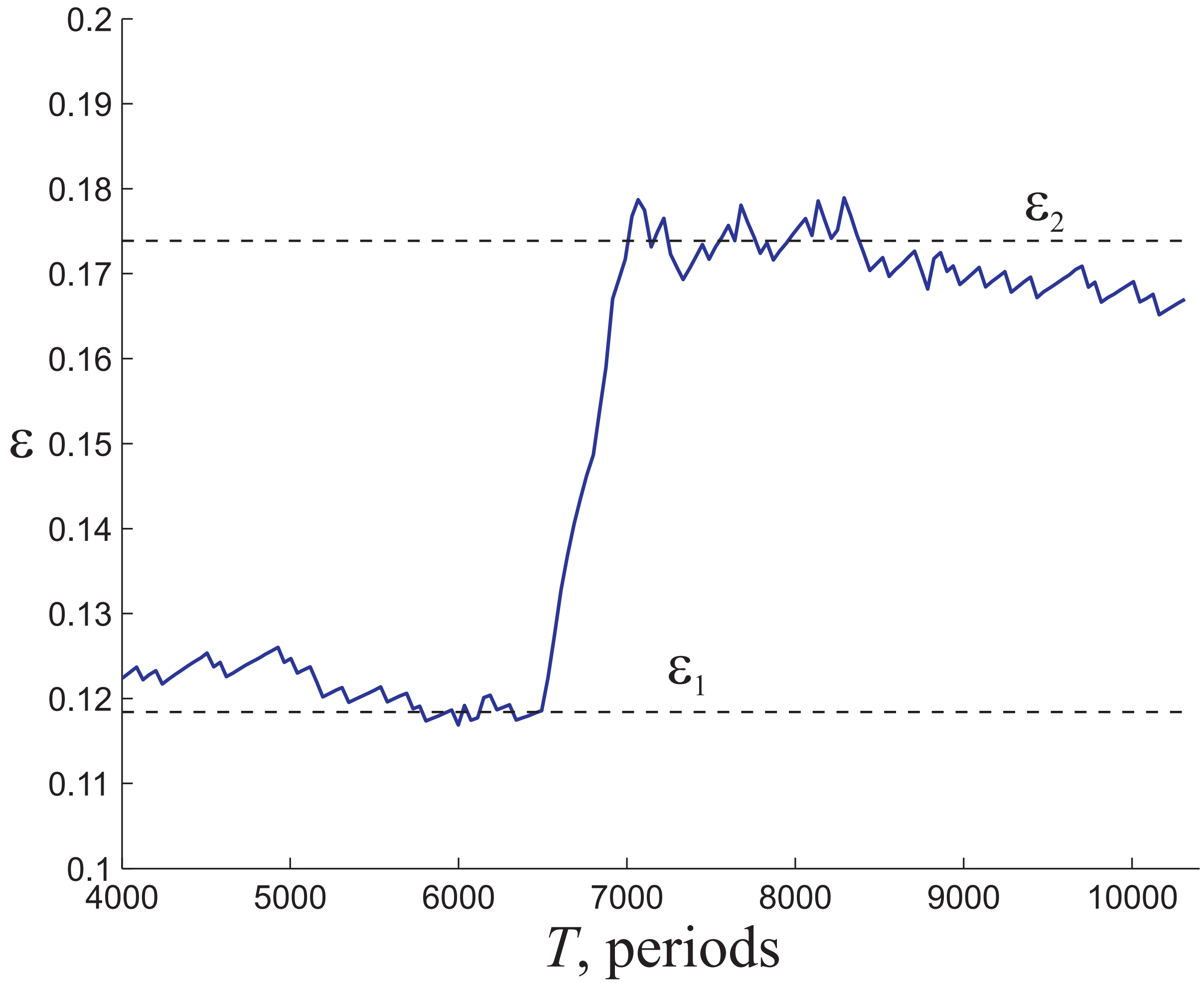


Forcing in (k, θ) space

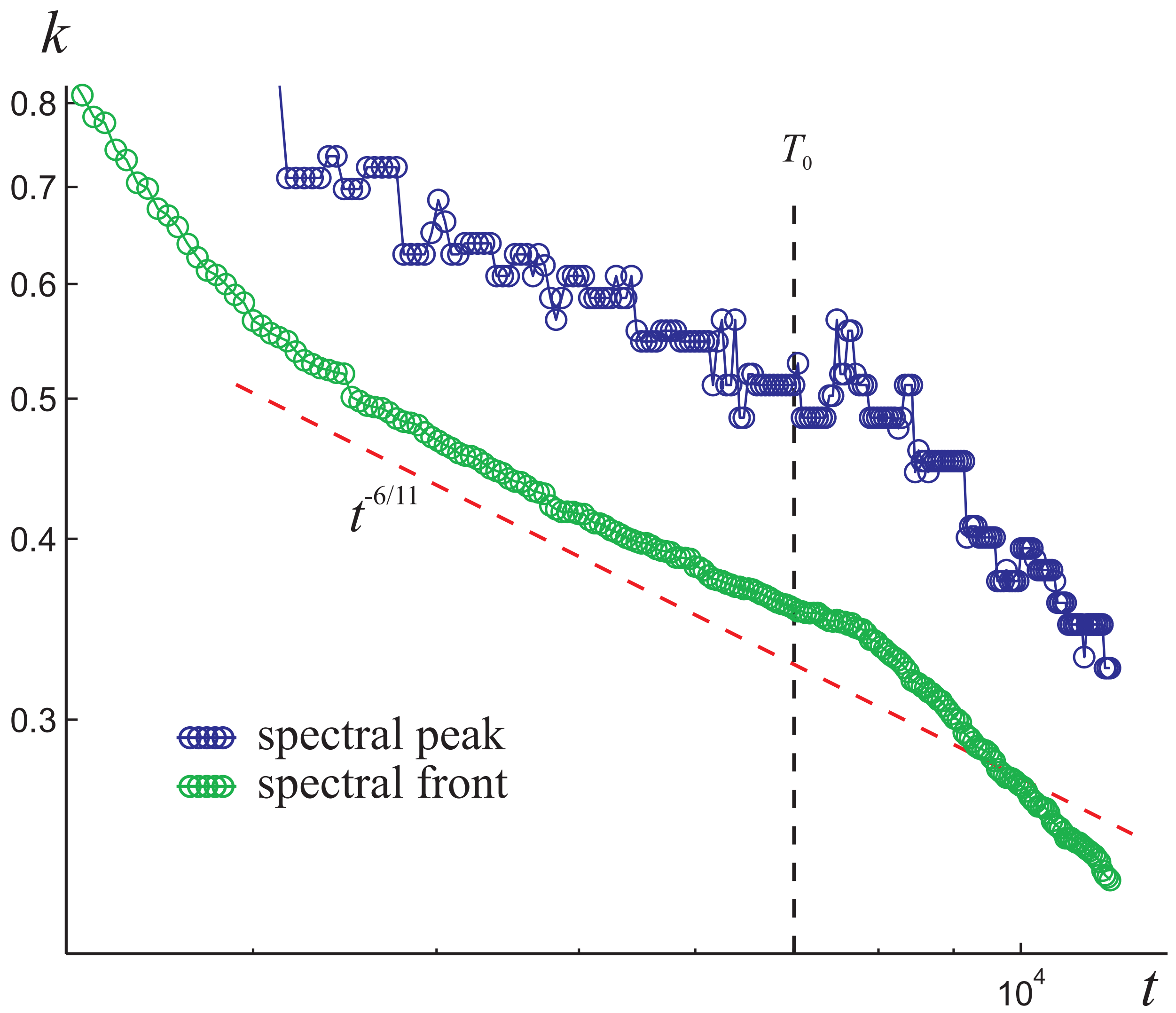


An example of DNS of wave action spectra evolution under abrupt increase of wind from 10 to 16 m/s. Curves are drawn in steps of 400 periods of the spectral peak.‘Equilibrium’ spectra $N_0(k)$ and $N_f(k)$ are shown by red curves.

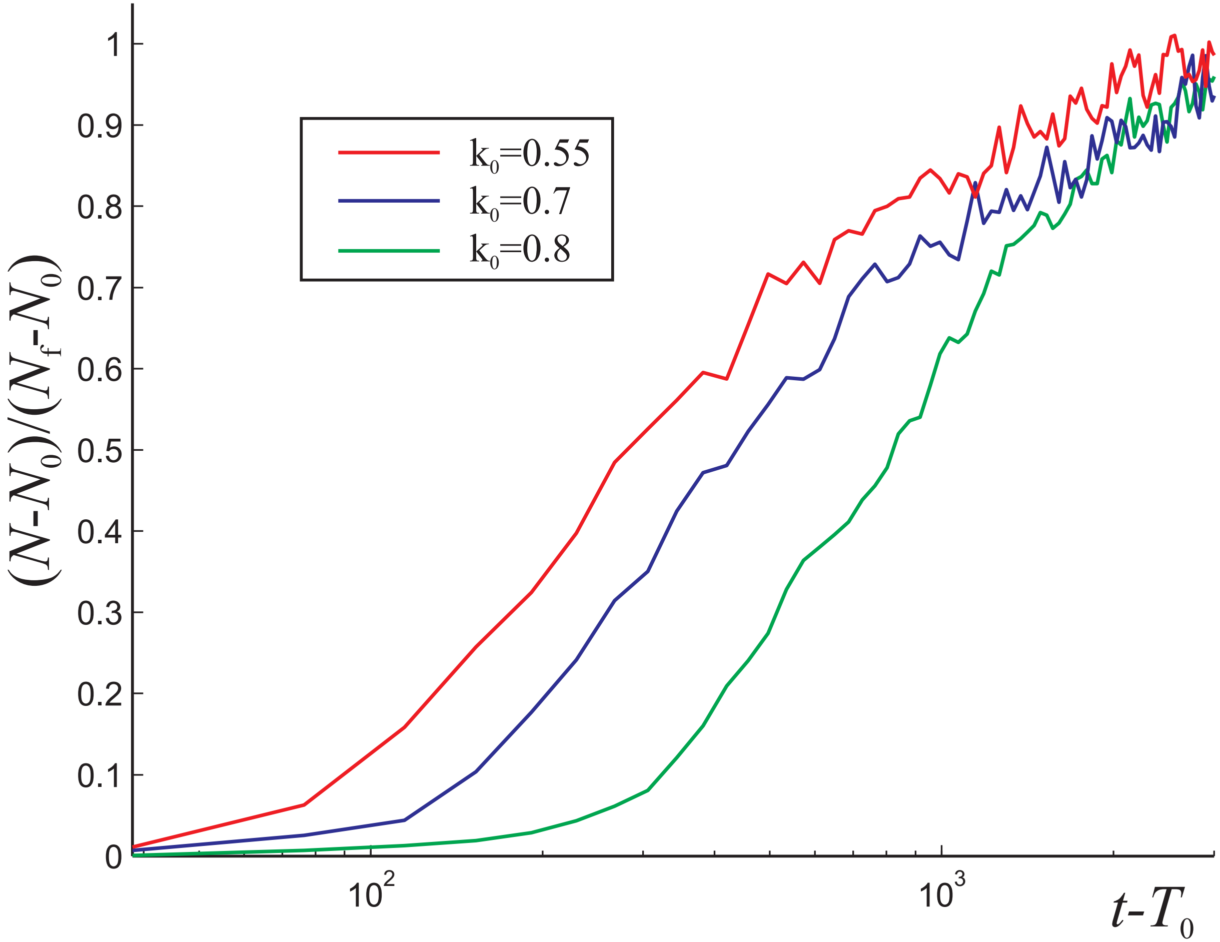
Evolution of wave steepness



Spectral peak and front

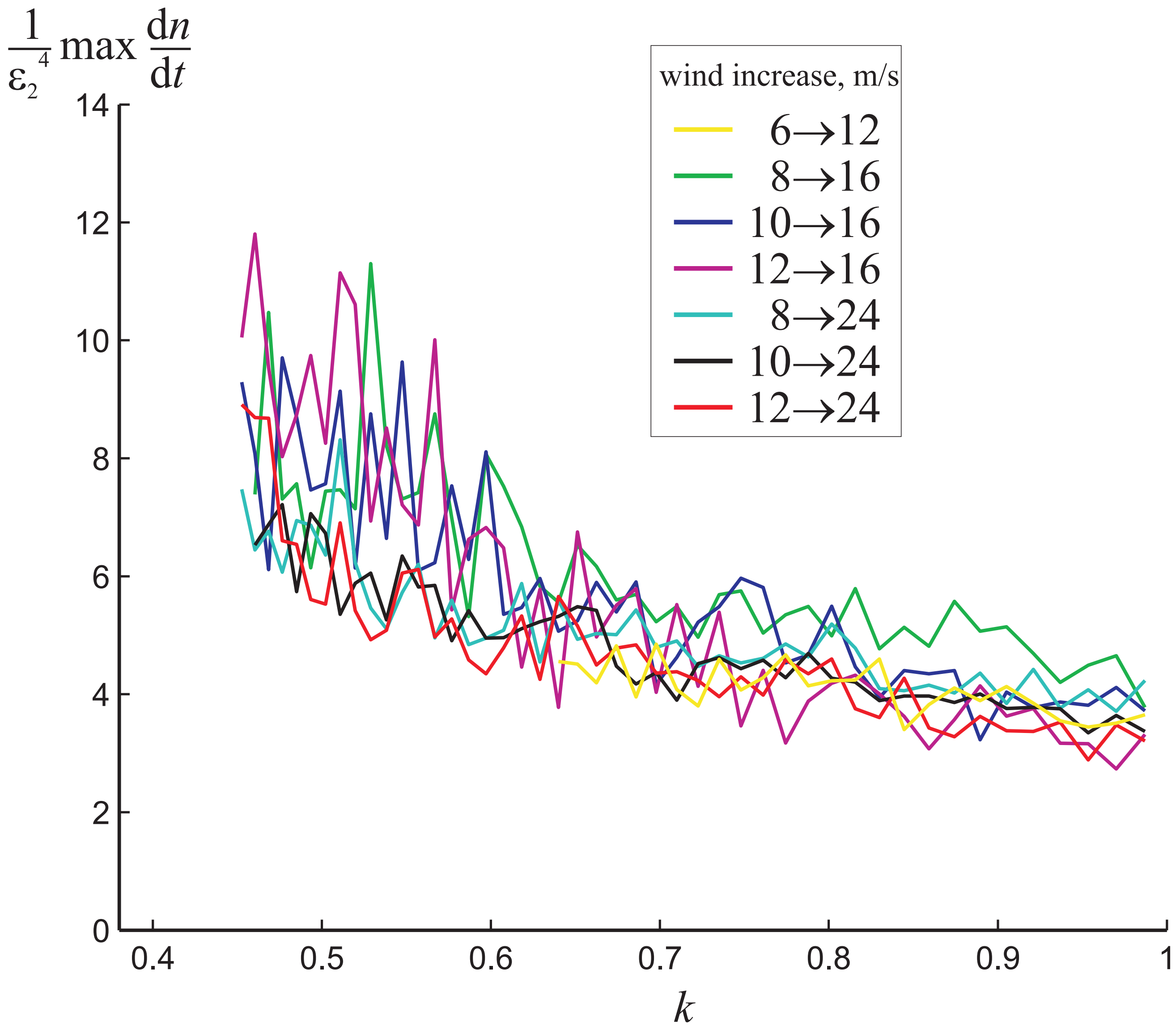


Growth on spectral slope

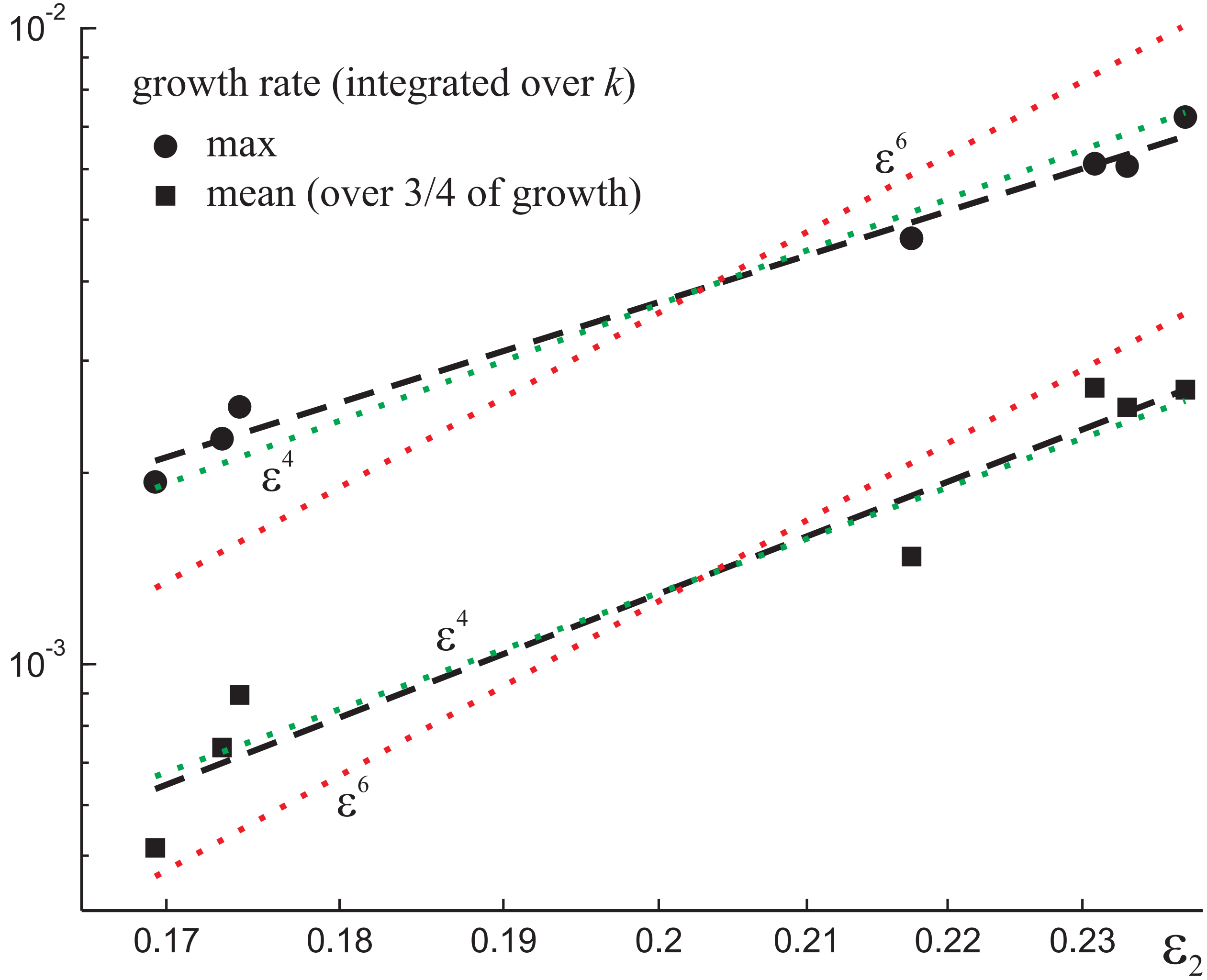


Samples of normalised growth at various k on the spectral slope

Growth rates on spectral slope



Figs/Growth rates on spectral slope



Growth rates

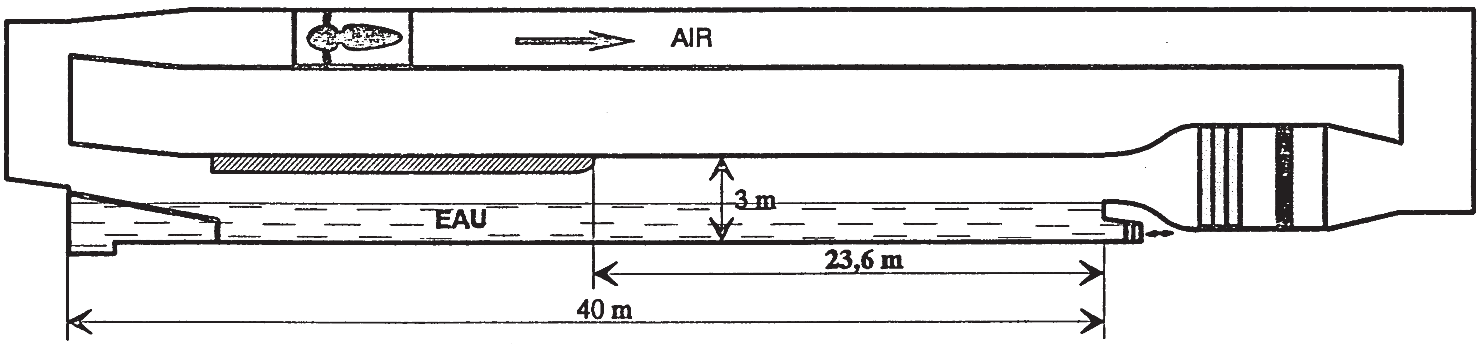
For growth rates, kinetic equation predicts a strict ε^6 scaling, equivalent to $O(\varepsilon^{-4})$ evolution timescale, provided that the growth is caused by the nonlinearity of the wave field, rather than directly by wind.

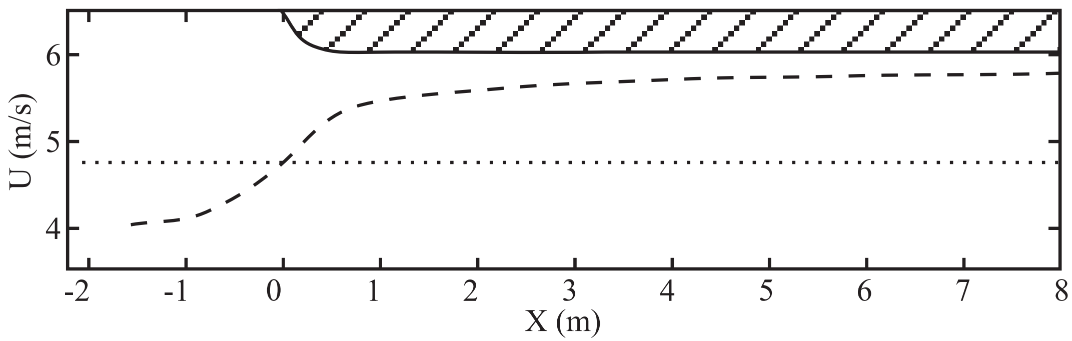
In our DNS, we observe the transition from a state where waves were in a local equilibrium with the wind, to a state when they are again in a local equilibrium with the new wind, so that growth rates are small at the start and finish of the transition, and attain maximum at a certain time.

These maximum growth rates (as well as average growth rates) scale as ε^4 (equivalent to $O(\varepsilon^{-2})$ evolution timescale). Here ε refers to the new 'equilibrium', and does not depend on the 'old' conditions.

Experiments on rapid wind increase

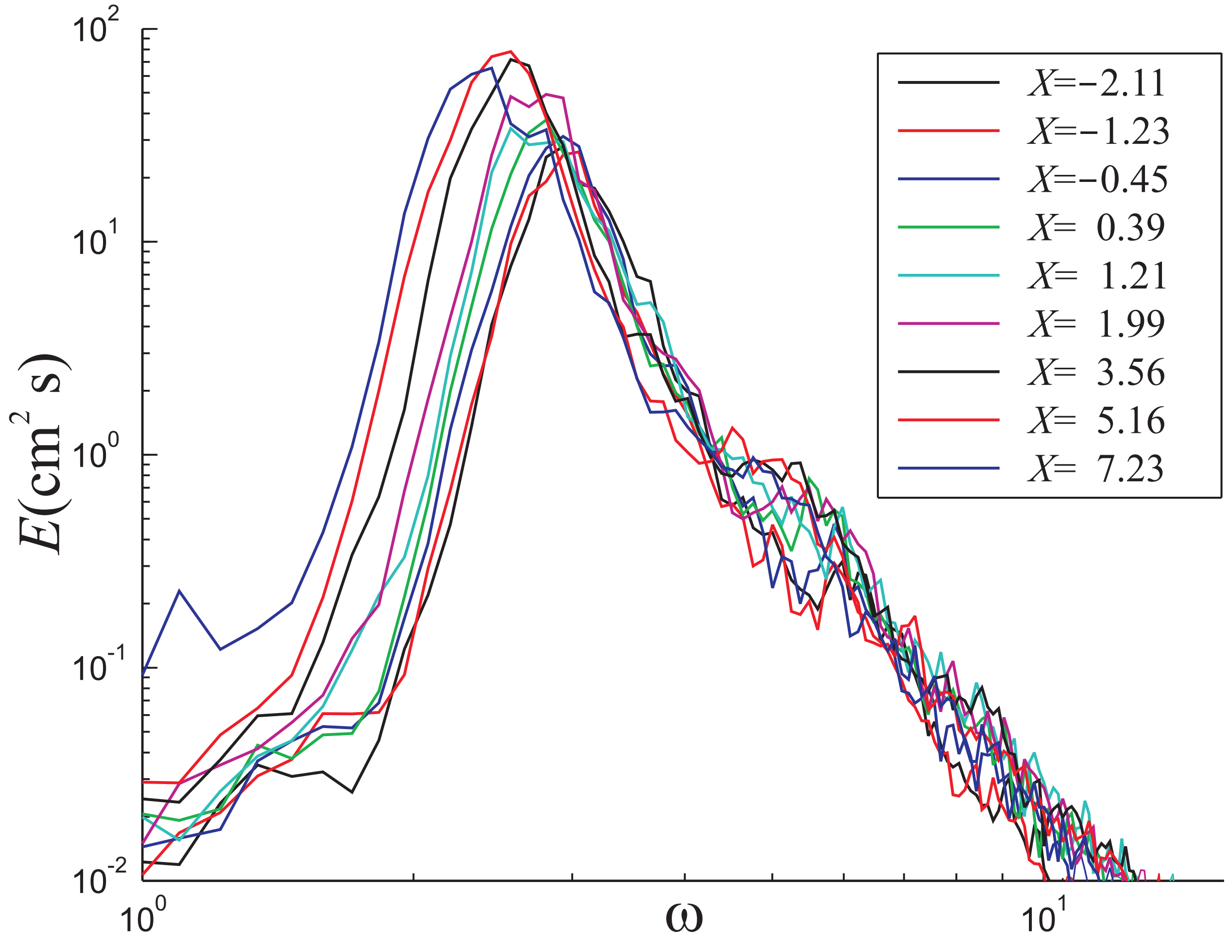
by Guillemette Caulliez & Laurent Autard (Marseille)





Wind with initial speed of 3–6 m/s is increased by 50% at relative fetch 0, by a false ceiling

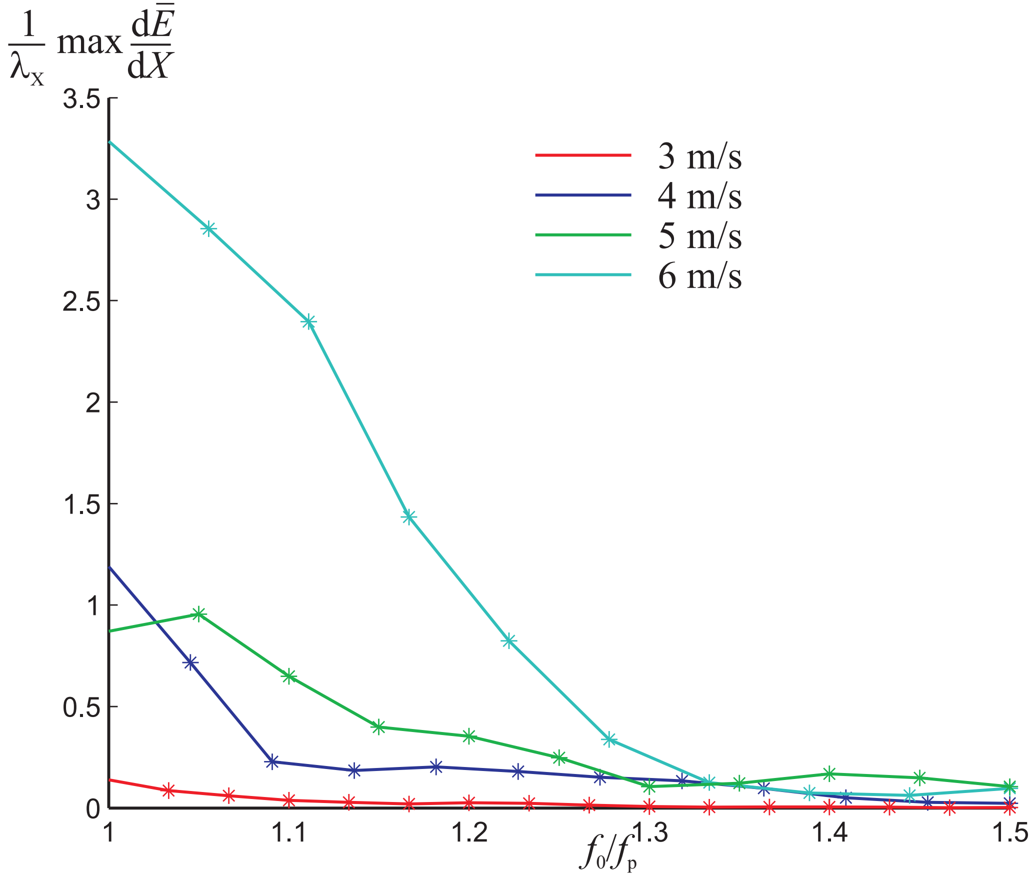
Energy spectrum

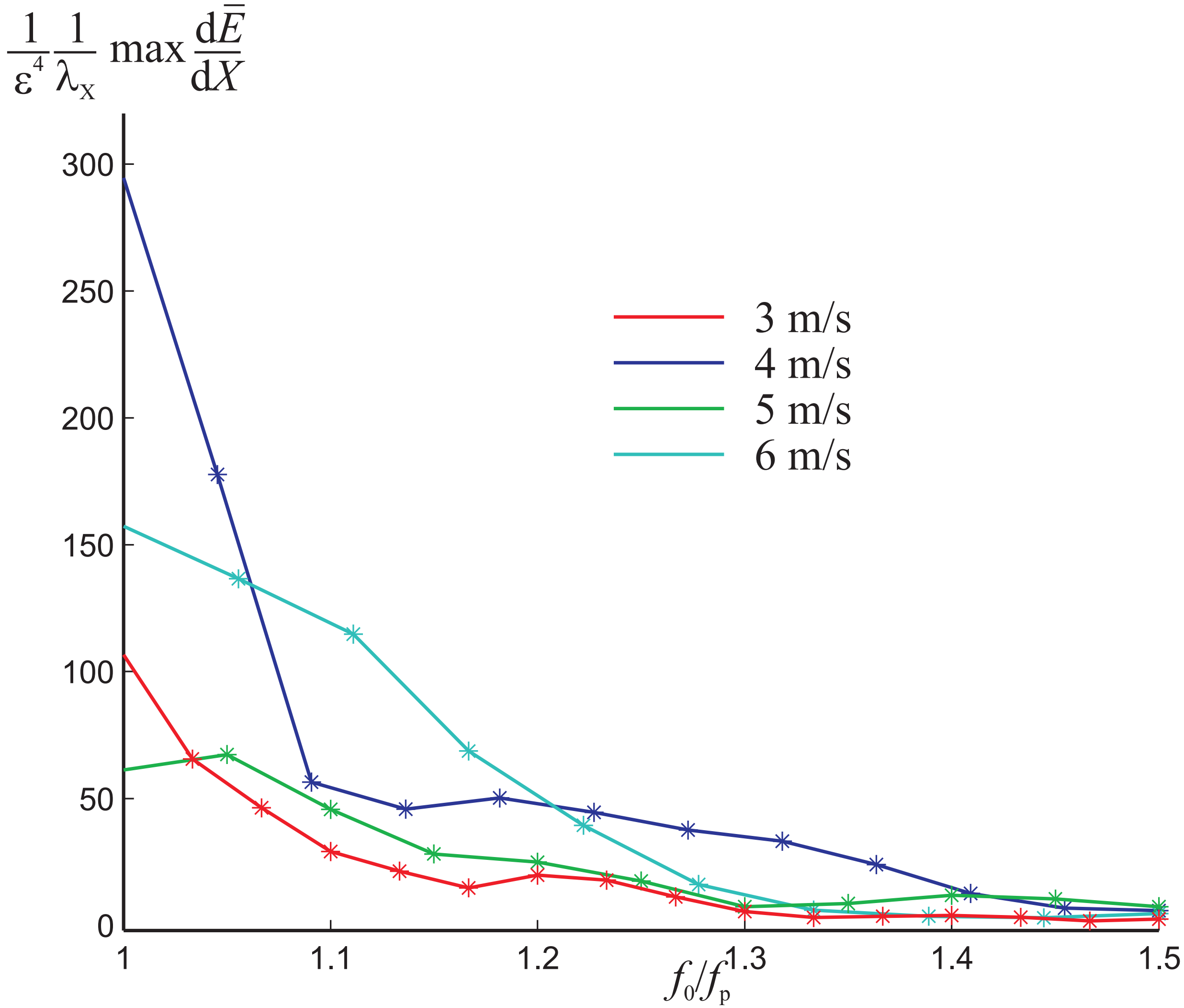


Evolution of energy spectrum with fetch, for $U = 4$ m/s

Growth rates

Maximal growth rates in the range $f_p < f < 2f_p$ (f_p is the spectral peak frequency). Under $O(\varepsilon^4)$ normalization the growth rates nearly collapse onto a single curve.





$\frac{1}{\varepsilon^4}\frac{1}{\lambda_x}\max\frac{d\bar{E}}{dX}$
300
250
200
150
100
50
0
1
1.1
1.2
1.3
1.4
1.5
f_0/f_p
3 m/s
4 m/s
5 m/s
6 m/s

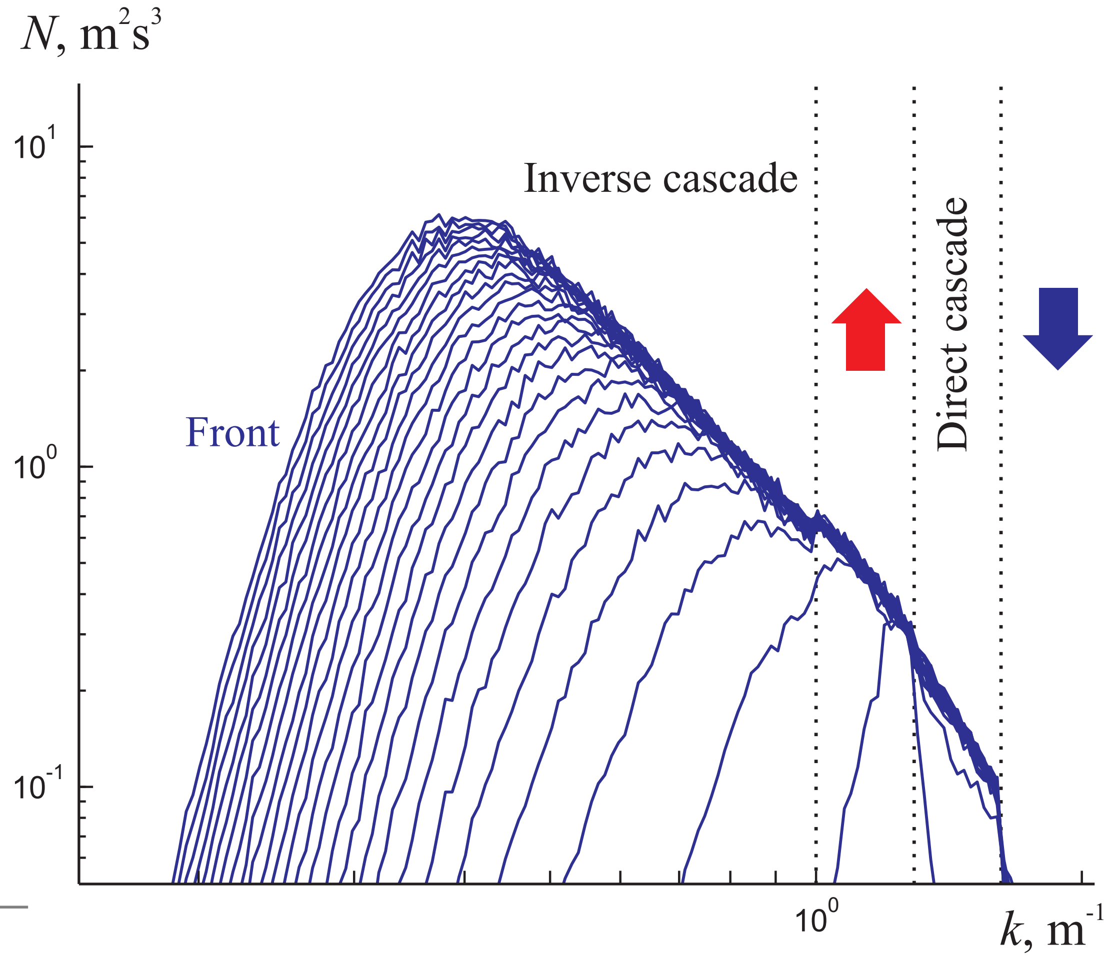
Intermediate Conclusions

- By examining an example of a very common "perturbed" situation we showed that the evolution of wave spectra occurs on the ‘dynamic’ $O(\varepsilon^{-2})$ timescale, rather then the ‘kinetic’ $O(\varepsilon^{-4})$ one.
- after an instant increase of forcing, growth rates of spectra scale as ε^4, not ε^6, where ε refers to the state of equilibrium with forcing *after* the increase of forcing.
- the phenomenon is not confined to to wind waves, and is relevant to the general case of sharp external perturbation of a random wave field.
- for wind waves, the results show that modelling of all situations when a wave field is driven out of the local equilibrium (e.g. currents, internal waves, obstacles, variable forcing, etc) should be radically revised.

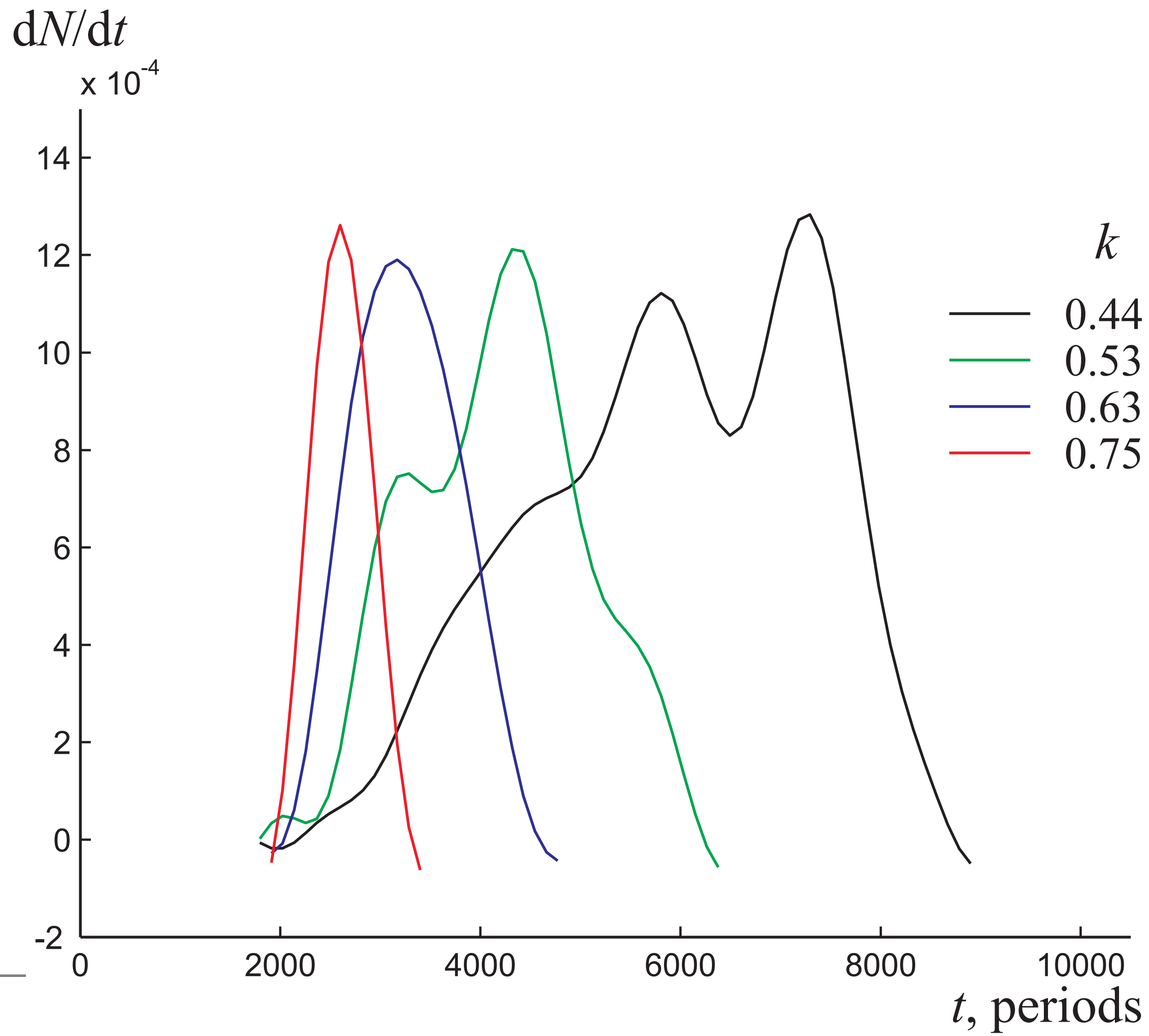
Everything is uniform and steady

NS time evolution of wave action spectrum $N($

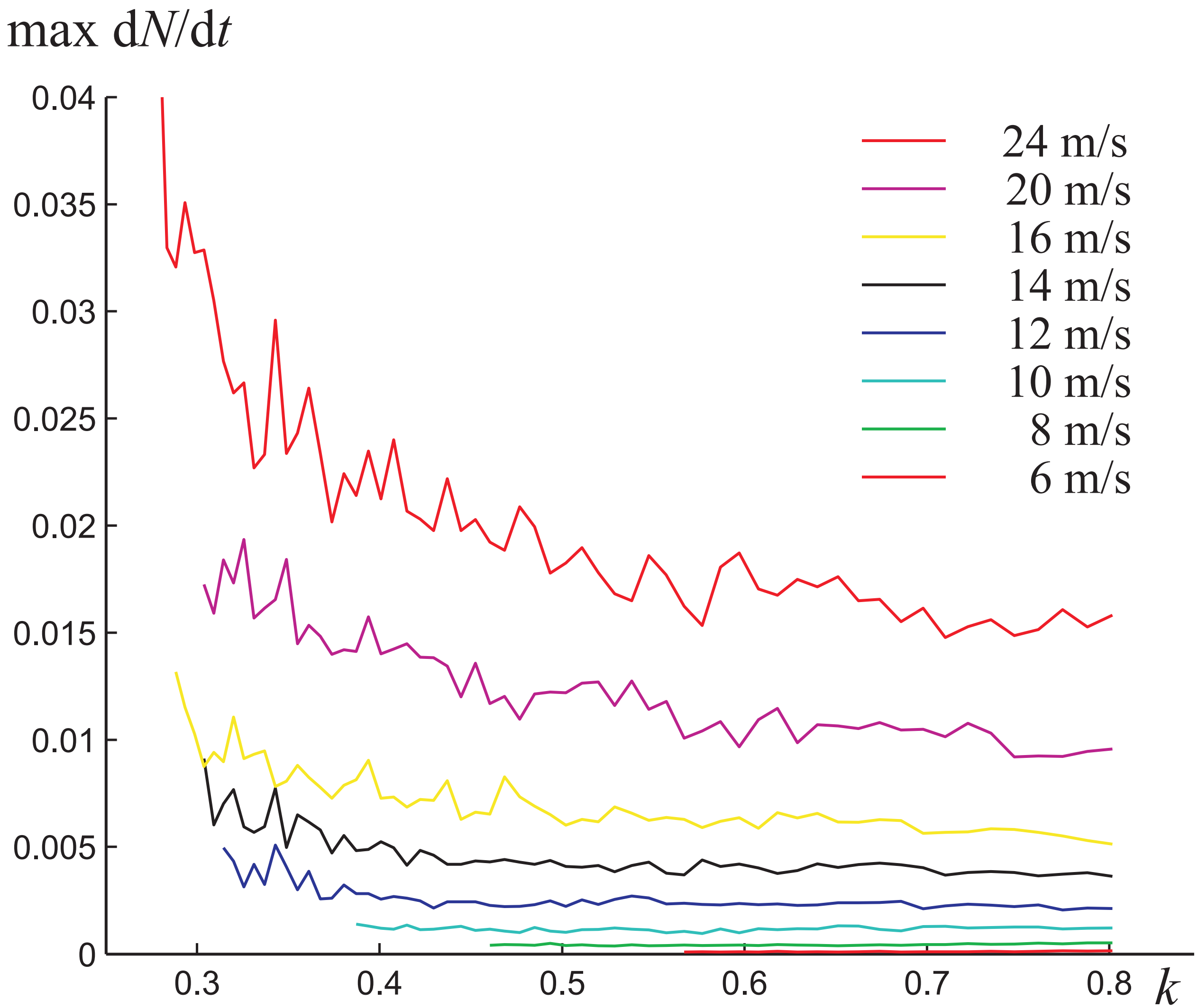
Time evolution of wave action spectrum $N(k)$ for a constant wind speed 10 m/s averaged over 30 realisations. Spectra are drawn in steps of 400 periods of the spectral peak.



Rate of wave action increase $\mathrm{d}N/\mathrm{d}t$ vs time (smoothed by a spline) during the passing of the spectral front for different wavenumbers, for wind speed 10 m/s.



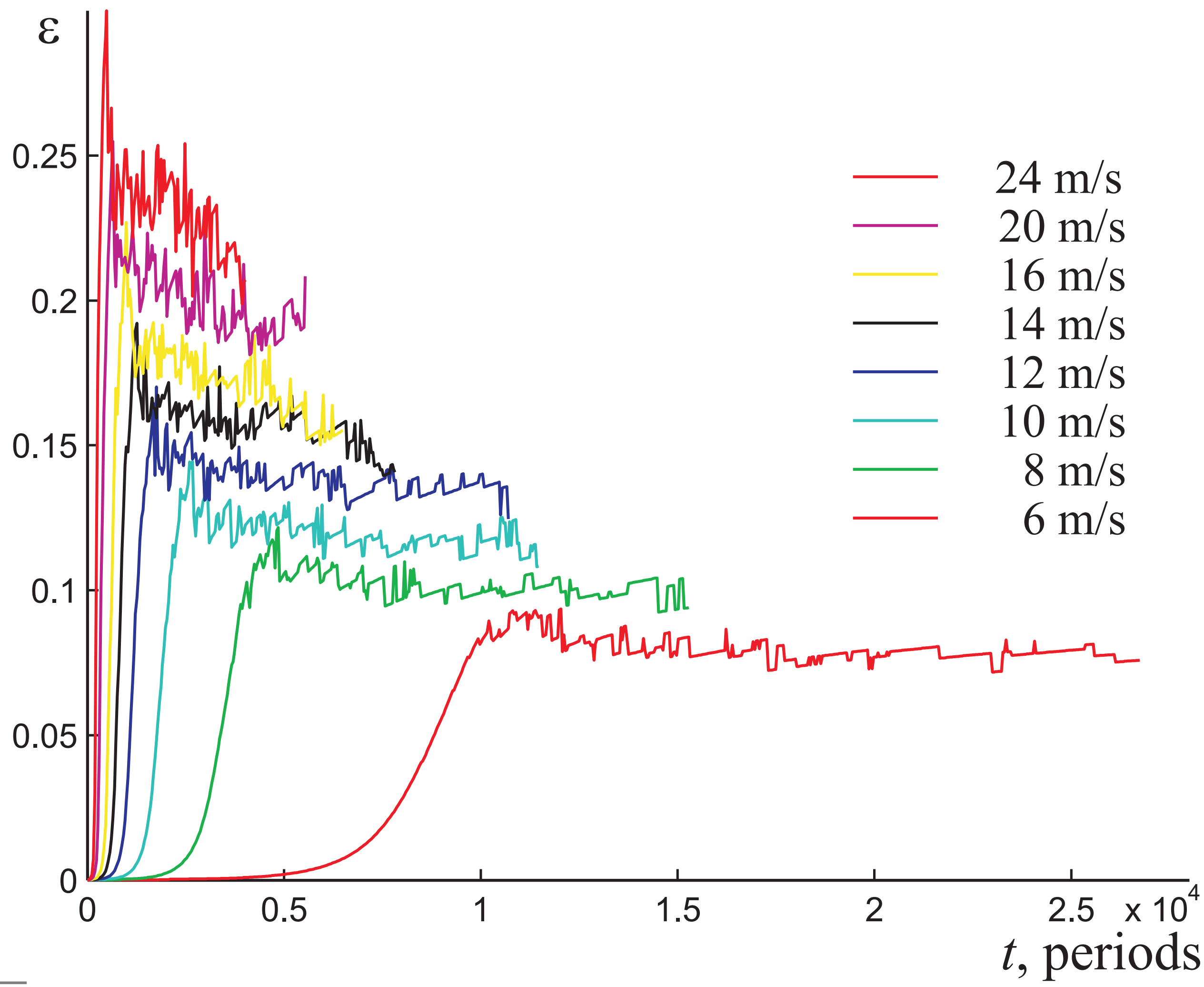
The maxima of $\mathrm{d}N/\mathrm{d}t$ vs k for different wind speeds in the range 6–24 m/s.



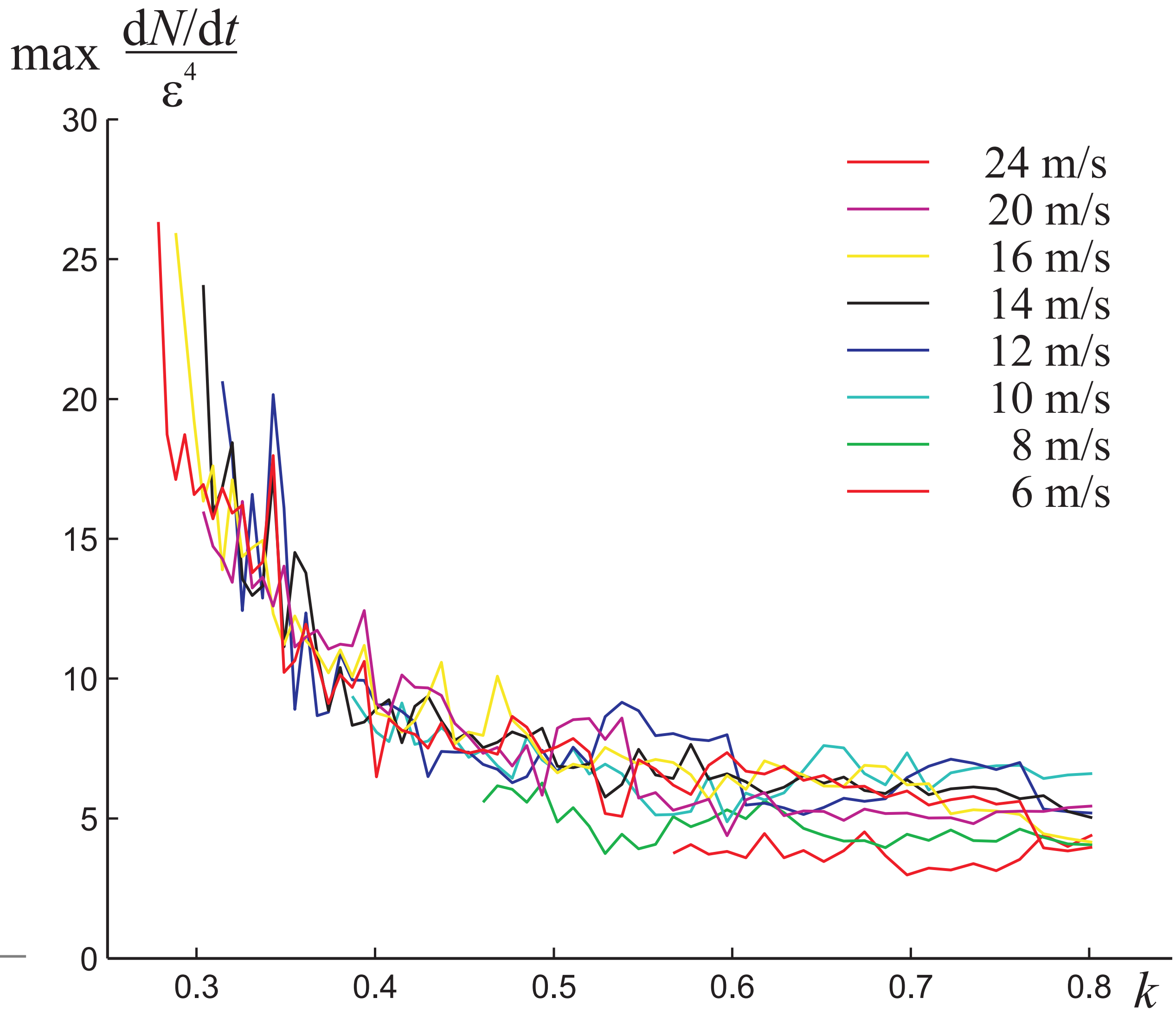
Evolution of steepness

Wave steepness ε vs time for different wind speeds

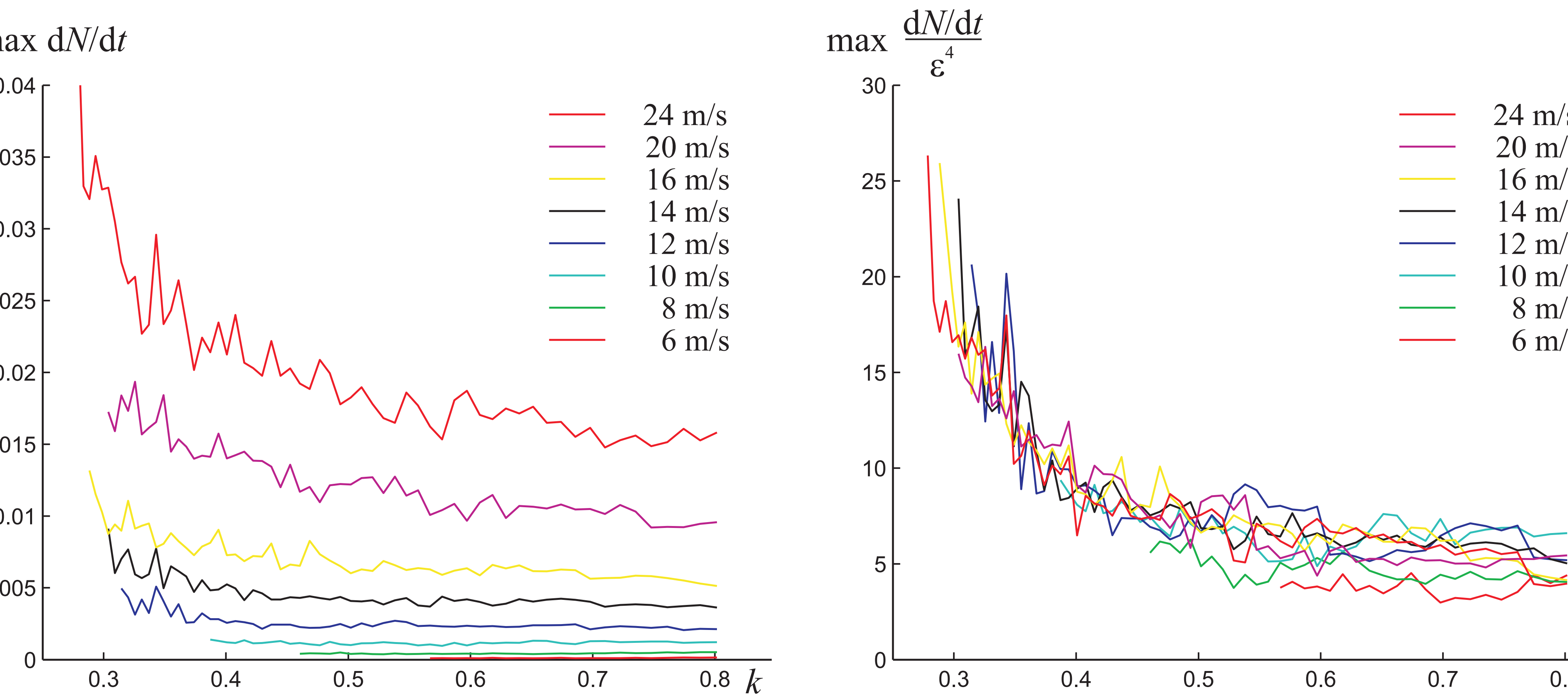
($\varepsilon = \sqrt{E}k_p/2\pi$, E is the total wave energy)



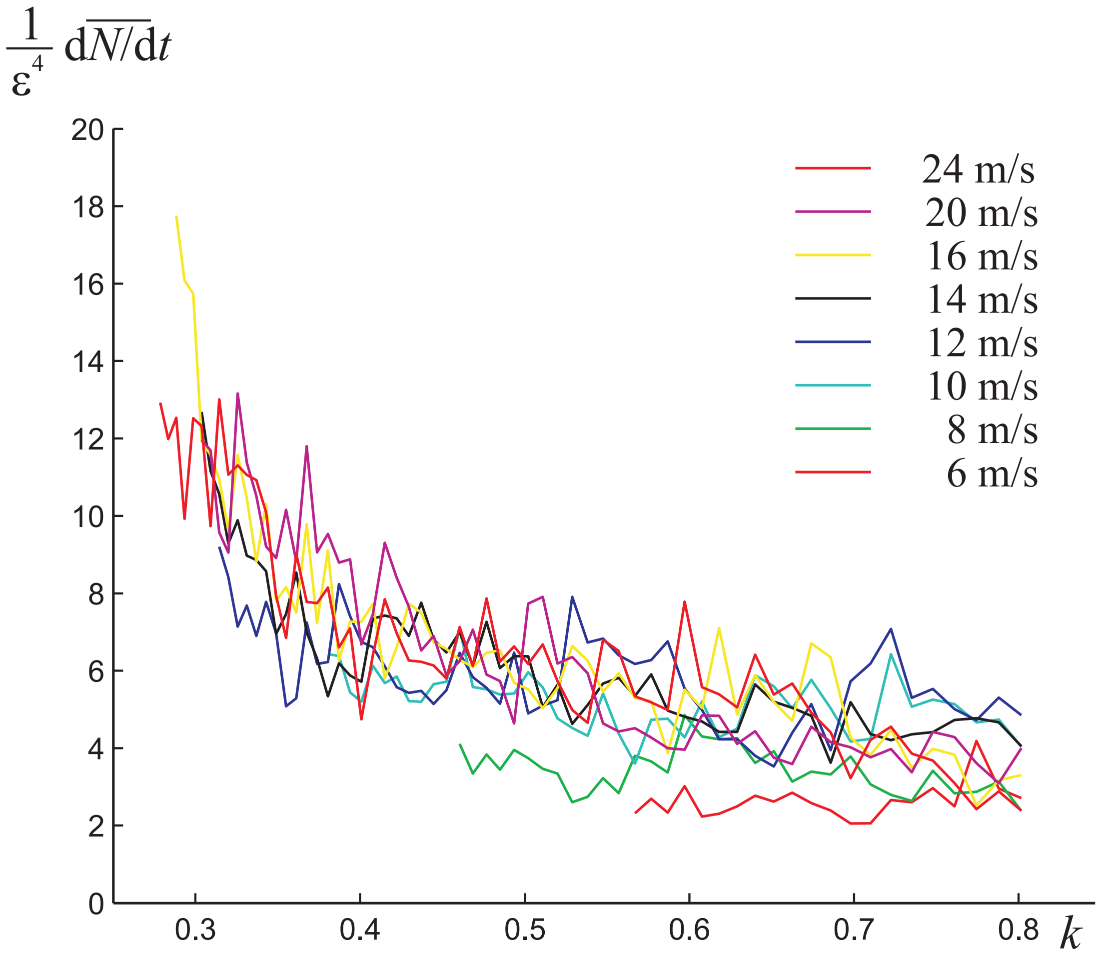
The maxima of $\mathrm{d}N/\mathrm{d}t$ vs k for different wind speeds, normalised by ε^4 (wave steepness at the passing of the spectral front for a given k)



Normalised vs non-Normalised dependencies



Mean values of $\mathrm{d}N/\mathrm{d}t$ (over the increase of $N(k)$ from $0.5N_m$ to N_m, where N_m is the maximum of $N(k)$ at the passing of the spectral front) vs k for different wind speeds, normalised by ε^4



Thus, at the front of the spectra nonlinear evolution occurs on the dynamic (ε^{-2}) and not on the kinetic (ε^{-4}) timescale.

Hence, the KE cannot *quantitatively* describe such an evolution.

Why? What could be done?

Why the KE description breaks down?

Recall the derivation of the KE.

The starting point of our analysis is the "reduced" Zakharov equation

$$\mathrm{i}\frac{\partial b_0}{\partial t} = \omega_0 b_0 + \int T_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123} + \dots$$

Notation: $\delta_{0+1-2-3} = \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, $\mathrm{d}\mathbf{k}_{123} = \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \mathrm{d}\mathbf{k}_3$.

Statistical description

Now we consider ensembles of random wave fields (each being governed by the deterministic Zakharov eq-n). We are interested in the ensemble averaged characteristics of the wave field.

Assumption of spatial homogeneity yields

$$< b_0^* b_1 >= n_0 \delta_{0-1}$$

brackets mean ensemble averaging, the second-order correlator, n_0 is the spectral density of wave action at wavevector $\mathbf{k} = \mathbf{k}_0$.

The classical problem is to find (and solve) a closed equation in terms of $n(k)$, i.e. to find evolution of wave action spectral density $n(k)$ with time.

The key steps and assumptions

On multiplying the reduced Zakharov equation by b_0^*, and its c.c. by b_0, upon ensemble averaging we immediately find

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im} \int T_{0123} J_{0123} \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}$$

$$J_{0123}^{(0)} \delta_{0+1-2-3} = < b_0^* b_1^* b_2 b_3 >$$

Assumption of Gaussianity yields

$$< b_0^* b_1^* b_2 b_3 >= n_0 n_1 \left(\delta_{0-2} \delta_{1-3} + \delta_{0-3} \delta_{1-2} \right) .$$

which is a real quantity and, since T_{0123} is also real, does not contribute to evolution of n_0.

Completely random phases provide no spectral evolution!

Find non-gaussian correction $J^{(1)}_{0123}$.

$J^{(1)}_{0123}$ is specified by an evolution equation containing on the right-hand-side the sixth-order correlator I_{012345}.

By assuming quasi-Gaussianity $I^{(0)}_{012345}$ is expressed in terms of the products of pair correlators. As a result we have

$$\left(\mathrm{i}\frac{\partial}{\partial t}+\Delta\omega\right)J^{(1)}_{0123}=2T_{0123}f_{0123}, \tag{1}$$

where $\Delta\omega=\omega_0+\omega_1-\omega_2-\omega_3$, and $f_{0123}=n_2n_3(n_0+n_1)-n_0n_1(n_2+n_3)$

The crucial assumption

It is usually assumed that n_0 and, hence, f_{0123} depends on slow time μt, such that $\mu/\Delta\omega \ll 1$.

Then neglecting $\frac{\partial}{\partial t}$ in $\left(\mathrm{i}\frac{\partial}{\partial t} + \Delta\omega\right) J^{(1)}_{0123} = 2T_{0123} f_{0123} \qquad \Rightarrow$

$$J^{(1)}_{0123}(t) \simeq \frac{2T_{0123}}{\Delta\omega} f_{0123}.$$

This solution represents a large t asymptotics and is understood in terms of generalized functions

$$J^{(1)}_{0123}(t) = 2T_{0123}\left[\frac{P}{\Delta\omega} + \mathrm{i}\pi\delta(\Delta\omega)\right] f_{0123}(t), \qquad \text{(P is ‘principal val}$$

This asymptotic derivation yields the classic kinetic equation and is valid as long as the interest is confined to slow $O(\varepsilon^{-4})$ evolution.

The generalised kinetic equation

If we allow for faster variability of statistical moments of wave field, we can use the exact solution for $J^{(1)}$ in the form

$$J^{(1)}_{0123}(t) = -2\mathrm{i}T_{0123}\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau - t)} f_{0123}\,\mathrm{d}\tau + J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}.$$

$J^{(1)}_{0123}(0)$ is specified by initial conditions.

The resulting "generalized" kinetic equation reads

$$\frac{\partial n_0}{\partial t} = 4\mathrm{Re}\int T^2_{0123}\left[\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau - t)} f_{0123}\,\mathrm{d}\tau\right]\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}$$

$$+2\mathrm{Re}\int\left[\mathrm{i}T_{0123}J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}\right]\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}.$$

The GKE describes ε^{-2} evolution and tends to the classic KE at large times.

GKE

Since the GKE tends to the classic KE at large times, the stationary solutions of the KE are also equilibrium solutions of the GKE.

Initial stages

In general setting the evolution of spectral density n depends not only on the initial distribution of n, but also on the initial distribution of $J^{(1)}_{0123}(0)$.

"Cold start". Zero value of $J^{(1)}_{0123}(0)$ corresponds to the situations where the wave field is initially free, so that the wave components are not correlated, and waves begin to interact only after $t = 0$. Then the GKE reads

$$\frac{\partial n_0}{\partial t} = 2 \int T^2_{0123} \left[\int_0^t \cos[\Delta\omega(\tau - t)] f_{0123} \, \mathrm{d}\tau \right] \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}$$

$$\frac{\partial n_0}{\partial t}\Big|_{t=0} = 0, \qquad \frac{\partial^2 n_0}{\partial t^2}\Big|_{t=0} = 2 \int T^2_{0123} f_{0123} \, \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}$$

All odd derivatives are zero, all even derivatives are known.

The Timescales

$$\frac{\partial n_0}{\partial t}\Big|_{t=0} = 0, \qquad \frac{\partial^2 n_0}{\partial t^2}\Big|_{t=0} = 2\int T_{0123}^2 f_{0123}\,\mathrm{d}\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}$$

Since $n \sim \varepsilon^2$ and the RHS is $\sim n^3 \sim \varepsilon^6$, then the timescale of initial evolution is $O(\varepsilon^{-2})$.

Why the fast evolution?

If there is an imposed external perturbation (sharp change of wind, nonuniform current, etc) then the faster scales come from the outside and trigger nonlinear wave field response on the $O(\varepsilon^{-2})$ timescale.

When everything is strictly uniform. However, since we consider the inverse cascade regime

$$\frac{\partial n(\mathbf{k},\mathbf{t})}{\partial t}\frac{1}{\omega_p} \sim t^{\alpha_1} \qquad \frac{\partial n(\mathbf{k},\mathbf{t})}{\partial t}\frac{1}{n\omega_p} \sim t^{\alpha_2} \qquad \alpha_1, \alpha_2 > 0$$

That is while the peak frequency decreases, both the absolute and normalised $\partial n/\partial t$ grow with time, which invalidates the use of long time asymptotics employed in the derivation of the KE.

An alternative explanation not linked to the KE:
if we consider a pair of harmonics interacting with many other pairs of much smaller amplitude. Only those pairs which happen to be in phase grow fast and therefore make a noticeable contribution to wave field.

Conclusions

- Fast ε^{-2} evolution of wave spectra occurs even if everything is strictly uniform as an intrinsic property of the wave field dynamics.
- Moreover most of the change occurs on the fast scale.
- The phenomenon is not confined to water waves and the situations of inverse cascade.