About Shape of Freakon

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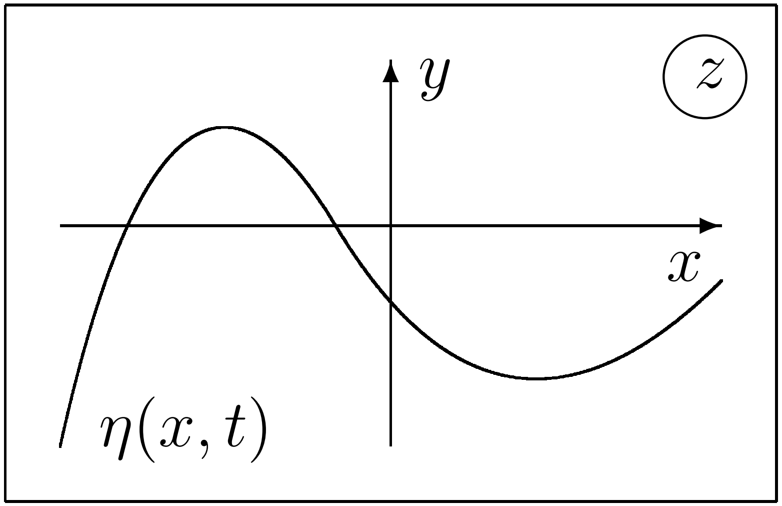
Potential Flow of 2D Ideal Fluid



irrotational

$$\triangle\phi(x,y,t)=0$$

Boundary conditions: $$\left[\begin{array}{l} \frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=\frac{P}{\rho}, \\ \frac{\partial\eta}{\partial t}+\eta_x\phi_x=\phi_y \end{array}\right| \text{ at } y=\eta(x,t).$$

$$\begin{aligned} \frac{\partial\phi}{\partial y} &= 0, y\to-\infty, \\ \frac{\partial\phi}{\partial x} &= 0, |x|\to\infty, \text{or periodic} \end{aligned}$$

NLSE approximation

From the equation for potential flow

$$\begin{aligned} \frac{\partial \phi}{\partial t} + \frac{1}{2}\phi_x^2 + g\eta &= -\frac{P}{\rho} \qquad \text{at } z = \eta, \\ \frac{\partial \eta}{\partial t} + \eta_x \phi_x &= \phi_z \qquad \text{at } z = \eta. \end{aligned} \tag{1}$$

one can derive nonlinear Shredinger equation:

$$i(\frac{\partial A}{\partial t} + C_g A_x) - \frac{\omega_0}{8k_0^2} A_{xx} - \frac{1}{2}\omega_0 k_0^2 |A|^2 A = 0. \tag{2}$$

A is the envelope of the surface elevation $\eta(x, t)$, so that

$$\eta(x, t) = \frac{1}{2}(A(x, t)e^{i(\omega_0 t - k_0 x)} + c.c.) \tag{3}$$

NLSE Soliton

Soliton solution for $A(x,t)$ is

$$A(x,t) = e^{-i\Lambda^2 t}\frac{\lambda}{\sqrt{2}k_0^2}\frac{\cos(k_0(x - V_{phase}t))}{\cosh(\lambda(x - C_g t))} \tag{4}$$

$$\Lambda^2 = \frac{\omega_0\lambda^2}{8k_0^2}.$$

Wavetrain of the amplitude a with wavenumber k_0 is unstable with respect to large scale modulation δk. Growth rate of the instability γ is

$$\gamma = \frac{\omega_0}{2}\left(\left(\frac{\delta k}{k_0}\right)^2 (ak_0)^2 - \frac{1}{4}\left(\frac{\delta k}{k_0}\right)^4\right)^{\frac{1}{2}}. \tag{5}$$

Here $\omega_0 = \sqrt{gk_0}$.

Conformal mapping

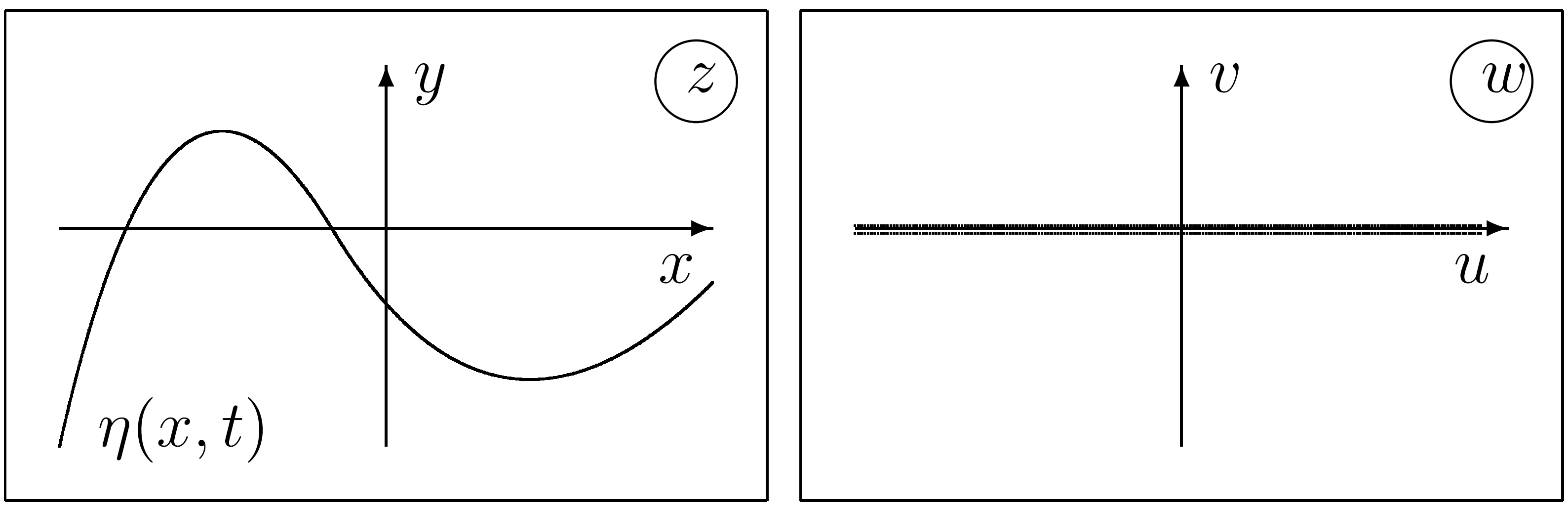
Domain on Z-plane $Z = x + iy$,

$$-\infty < x < \infty, \quad -\infty < y \leq \eta(x,t)\,,$$

to the lower half-plane,

$$-\infty < u < \infty, \quad -\infty < v \leq 0\,,$$

W-plane $W = u + iv$.



Equations for Z and Φ

If *conformal mapping* has been applied then it is naturally introduce complex analytic functions

$$Z = x + iy, \quad \text{and complex velocity potential} \quad \Phi = \Psi + i\hat{H}\Psi.$$

$$\begin{aligned} Z_t &= iUZ_u, \\ \Phi_t &= iU\Phi_u - \hat{P}\left(\frac{|\Phi_u|^2}{|Z_u|^2}\right) + ig(Z - u). \end{aligned}$$

U is a complex transport velocity:

$$U = \hat{P}\left(\frac{-\hat{H}\Psi_u}{|Z_u|^2}\right). \qquad\qquad u \to w$$

Projector operator $\hat{P}(f) = \frac{1}{2}(1 + i\hat{H})(f)$.

Cubic equations for R and V

Surface dynamics (and the fluid bulk!) is described by two analytic functions, $R(w,t)$ and $V(w,t)$. They are related to conformal mapping Z and complex velocity potential:

$$R = \frac{1}{Z_w}, \qquad \Phi_w = -iVZ_w.$$

For R and V dynamic equations have the simplest form:

$$\begin{aligned} R_t &= i\left[UR' - U'R\right], \\ V_t &= i\left[UV' - B'R\right] + g(R-1). \end{aligned}$$

Complex transport velocity U is defined as

$$U = \hat{P}(V\bar{R} + \bar{V}R), \qquad \text{and} \qquad B = \hat{P}(V\bar{V}).$$

NLSE and Dysthe and Conformal variables - I

Consider weakly nonlinear wave train. Use r instead of R

$$r = R - 1.$$

Then equations for R and V transform into

$$\begin{aligned} r_t + iV' &= i(-U' + Vr' - V'r + Ur' - rU'), \\ V_t - gr &= i(VV' - B' + UV' - rB'). \end{aligned} \tag{6}$$

$$U = \hat{P}(V\bar{r} + \bar{V}r). \tag{7}$$

NLSE and Dysthe and Conformal variables - II

We will look for the breather solution. It is periodic in some reference frame moving with velocity c. In this reference frame equations for r and V read

$$\begin{aligned} r_t - cr' + iV' &= i(-U' + Vr' - V'r + Ur' - rU') = F, \\ V_t - cV' - gr &= i(VV' - B' + UV' - rB') = G. \end{aligned}$$

We look for the solution of these equations in the following form

$$\begin{aligned} r &= \sum_{n=0}^{\infty} r_n(u,t)\mathrm{e}^{in(\Omega t - ku)}, \qquad k > 0 \\ V &= \sum_{n=0}^{\infty} V_n(u,t)\mathrm{e}^{in(\Omega t - ku)}. \end{aligned} \tag{8}$$

NLSE and Dysthe and Conformal variables - III

Thereafter we will put $k = 1$, $c = \frac{1}{2}$, $\Omega = \frac{1}{2}$. The leading terms in expansion (8) are

$$r_1, \qquad V_1 \sim \epsilon << 1.$$

Then

$$r_n \sim V_n \sim \epsilon^n, \quad r_0 \sim V_0 \sim \epsilon^3. \tag{9}$$

r_n, V_n are "slow" functions of u. In other words

$$\frac{r'_n}{r_n} \sim \frac{V'_n}{V_n} \sim \epsilon << 1. \tag{10}$$

For the slow componets (time derivatives)

$$\frac{\dot{r}_n}{r_n} \sim \frac{\dot{V}_n}{V_n} \sim \epsilon^2 << 1. \tag{11}$$

NLSE and Dysthe and Conformal variables - IV

To proceed in derivation of envelope equation we have to learn how to calculate projective operator of functions like $a(u)\mathrm{e}^{imu}$. Here $a(u)$ - any "slow" function of$u$.

$$\hat{P}(\mathrm{e}^{ikm}a(u)) = \begin{cases} 0, & m > 0, \\ \mathrm{e}^{ikm}a(u), & m < 0 \end{cases} \tag{12}$$

Only if $m = 0$, projection is a nontrivial operation.
Thereafter we put

$$V_1 = \epsilon\psi$$

and replace

$$\frac{\partial}{\partial u} \to \epsilon\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial t} \to \epsilon^2\frac{\partial}{\partial t}. \tag{13}$$

NLSE and Dysthe and Conformal variables - V

Using the rule (12) we find with accuracy up to ϵ^3

$$\begin{aligned} V_2 &= \epsilon^2(-i\psi^2 + \frac{\epsilon}{2}\psi\psi'), \\ r_2 &= \epsilon^2(\psi^2 + i\epsilon\psi\psi'), \\ r_0 &= i\epsilon^3\hat{P}(|\psi|^2)', \qquad V_0 = \epsilon^2\hat{P}(|\psi|^2)' \end{aligned} \tag{14}$$

r_1 and V_1 are related with relation

$$r_1 = V_1 - \frac{\epsilon}{2}V_1' \tag{15}$$

$$2i\dot{\psi} + \frac{1}{4}\psi'' + |\psi|^2\psi = \epsilon\left[\dot{\psi}' - \psi\hat{H}(|\psi|^2)' - 2i(|\psi|^2\psi)'\right] \tag{16}$$

This is the Dysthe equation in conformal variables. In the limit of $\epsilon \to 0$ it gives standart NLSE.

Stationary Solution - FREAKON

$$\psi = A(u)\mathrm{e}^{i\Phi}\mathrm{e}^{\frac{it}{2}}.$$

$A(u)$ and Φ - are real functions satisfying the equations

$$-A + \frac{1}{4}A'' + A^3 - \frac{1}{4}A\Phi'^2 = -\epsilon\left\{(\frac{1}{2} + 2A^2)\Phi' + A\hat{K}A^2\right\}. \quad (17)$$

$$\Phi' = \epsilon(1 - 6A^2). \quad (18)$$

Keeping in (17) terms of the order of ϵ^2 is exceeding of accuracy. Thus it can be simplified up to the form

$$-A + \frac{1}{4}A'' + A^3 + \epsilon A\hat{K}A^2 = 0. \quad (19)$$

$\hat{K}$ is pure negative, $\hat{K}\mathrm{e}^{iku} = -|k|\mathrm{e}^{iku}$.

Stationary Solution - FREAKON

equation (19) realize minimum of the functional

$$H = \int_{-\infty}^{\infty} \left\{ -\frac{1}{2}A^2 - \frac{1}{8}A'^2 + \frac{1}{4}A^4 + \frac{\epsilon}{4}A^2\hat{K}A^2 \right\}, \frac{\partial H}{\partial A} = 0. \quad (20)$$

Let us $A = \frac{a}{\cosh 2u}$. a - is still unknown value. As a result

$$H = -\frac{2}{3}a^2 + (\frac{1}{6} - 0.22\epsilon)a^4.$$

Condition $\frac{\partial H}{\partial A} = 0$ gives

$$a = \sqrt{\frac{2}{1 - 1.32\epsilon}}. \quad (21)$$

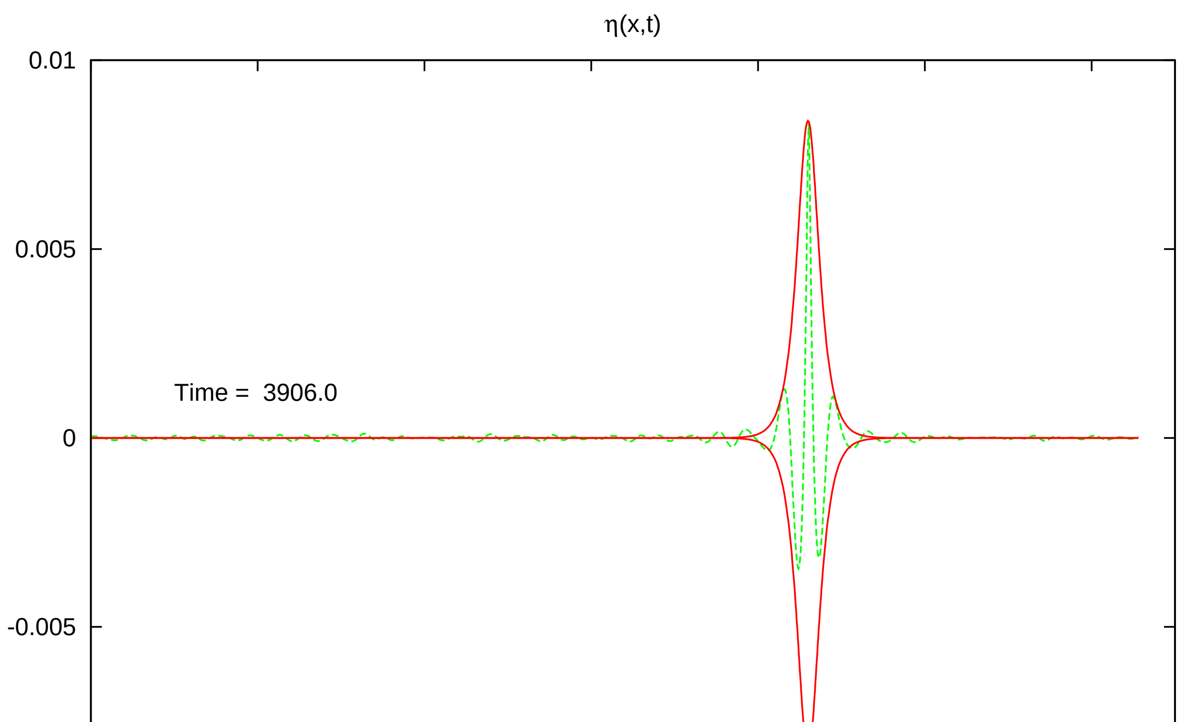
In the limit $\epsilon \to 0$ we get the NLSE result, $a = \sqrt{2}$. One can see that relatively small ϵ leads to the strong deviation from the NLSE limit.

NLSE SOLITON - FREAKON

We compare breather-type solution with the soliton shape. In the Figure 1 envelope is the following:

$$A = \frac{a}{\cosh \lambda x}.$$

with $a = 0.0084$, and $\lambda = 17$. If it were NLSE envelope with the same $\lambda = 17$, than a would be 0.0048.



NLSE Solition - FREAKON

NLSE solitons are lower and wider. This is in agreement with the theory.

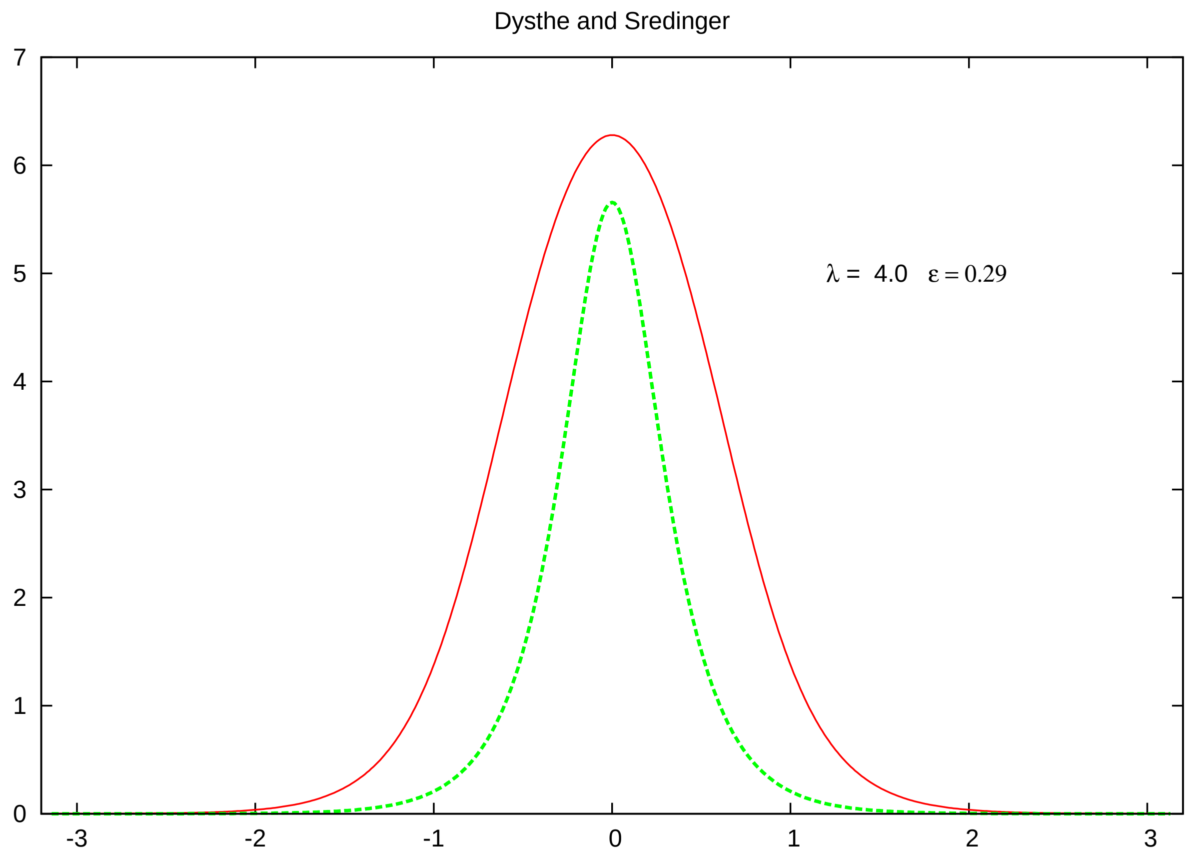


Figure 2. Solitons for NLSE and Dysthe equation. $\lambda = 4.0, \epsilon = 0.290$.

NLSE Solition - FREAKON

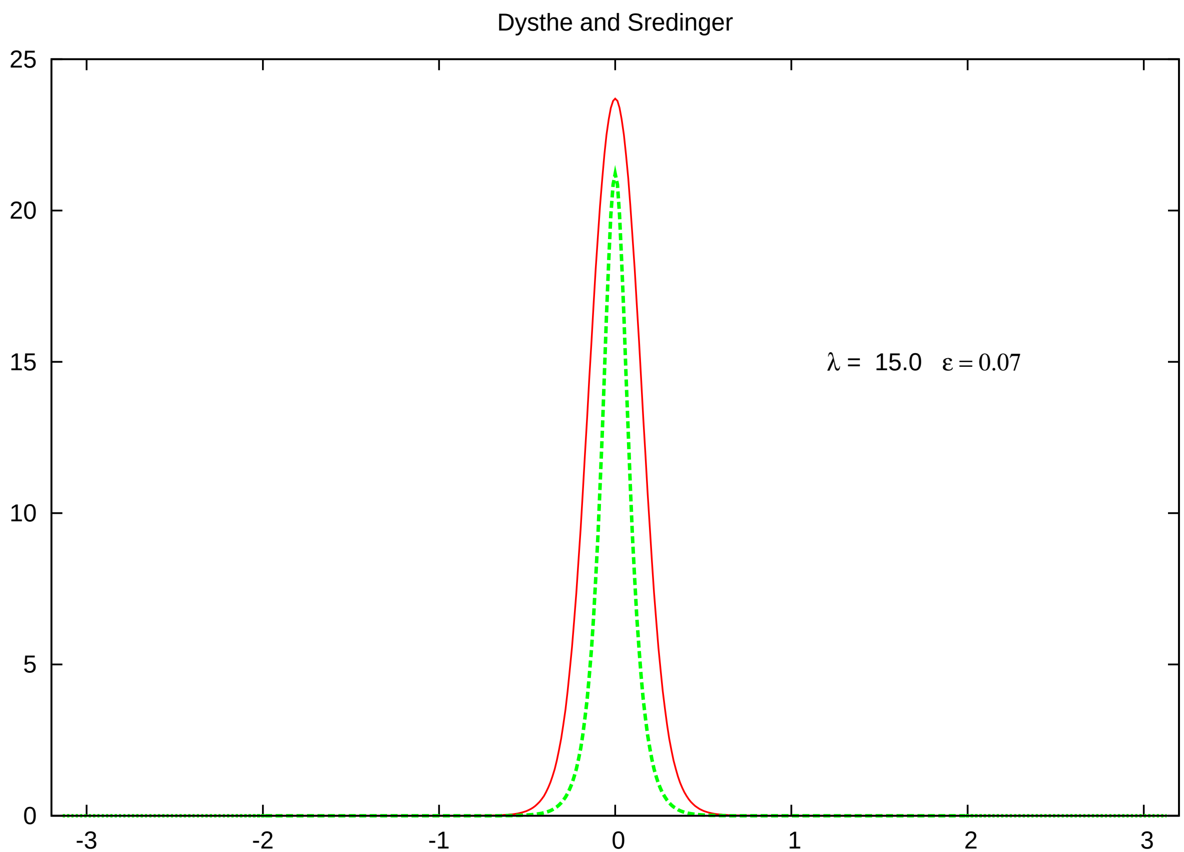


Figure 3. Solitons for NLSE and Dysthe equation. $\lambda = 15.0, \epsilon = 0.070.$

Giant Breather, k-ω spectrum

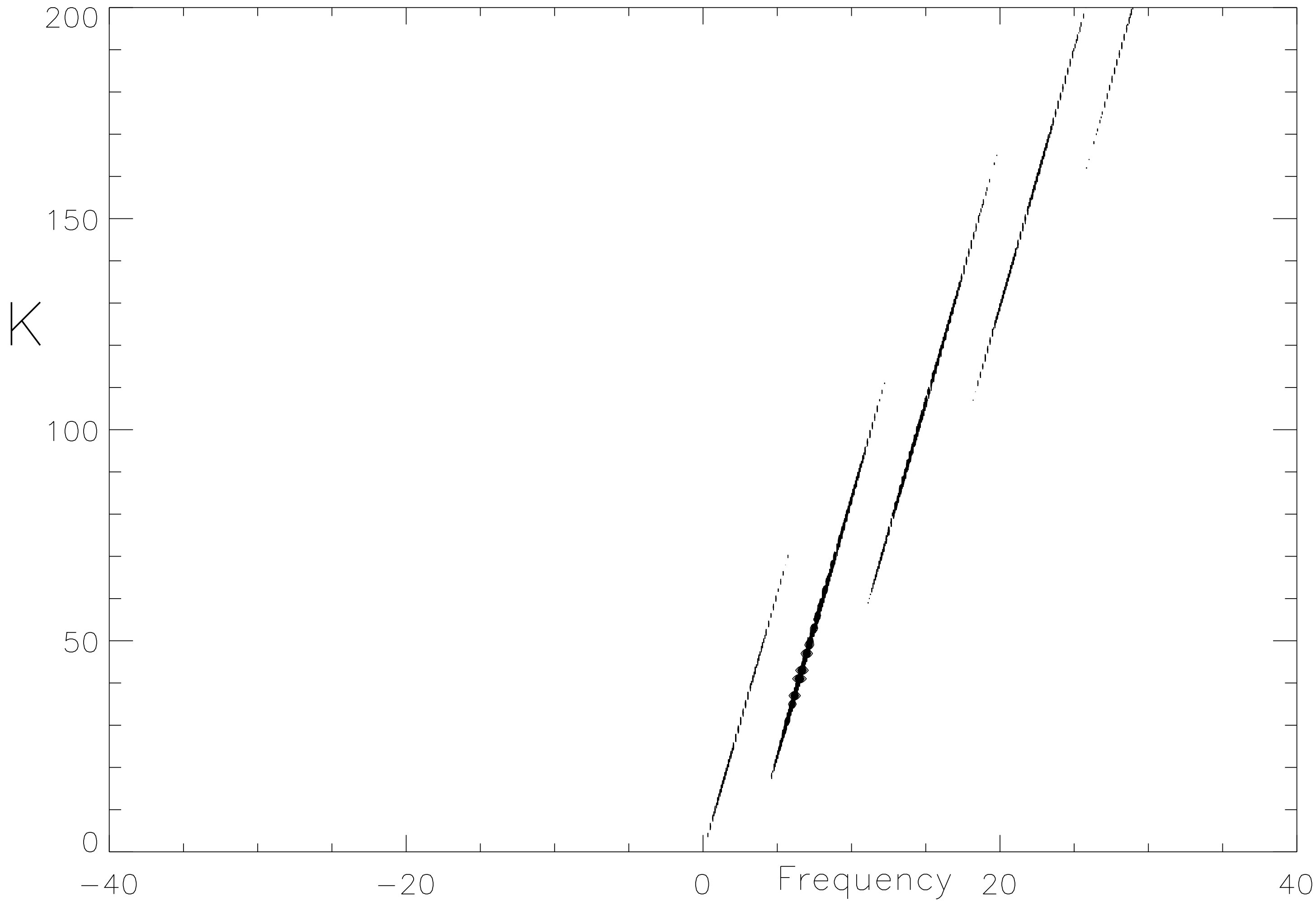


Figure 4. Negative frequency is absent!.