About Shape of Freakon

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Potential Flow of 2D Ideal Fluid

Boundary conditions:
\[
\begin{align*}
\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g\eta &= \frac{P}{\rho}, \\
\frac{\partial \eta}{\partial t} + \eta_x \phi_x &= \phi_y
\end{align*}
\]

\[
\left. \begin{array}{l}
\frac{\partial \phi}{\partial y} = 0, \quad y \to -\infty, \\
\frac{\partial \phi}{\partial x} = 0, \quad |x| \to \infty, \text{ or periodic}
\end{array} \right| \text{ at } y = \eta(x, t).
\]

Irrotational:
\[
\Delta \phi(x, y, t) = 0
\]
NLSE approximation

From the equation for potential flow

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} \phi_x^2 + g \eta = -\frac{P}{\rho} \quad \text{at } z = \eta, \]

\[ \frac{\partial \eta}{\partial t} + \eta_x \phi_x = \phi_z \quad \text{at } z = \eta. \]  \tag{1} \]

one can derive nonlinear Shredinger equation:

\[ i \left( \frac{\partial A}{\partial t} + C g A_x \right) - \frac{\omega_0}{8 k_0^2} A_{xx} - \frac{1}{2} \omega_0 k_0^2 |A|^2 A = 0. \]  \tag{2} \]

\( A \) is the envelope of the surface elevation \( \eta(x, t) \), so that

\[ \eta(x, t) = \frac{1}{2} (A(x, t)e^{i(\omega_0 t - k_0 x)} + c.c.) \]  \tag{3} \]
NLSE Soliton

Soliton solution for \( A(x, t) \) is

\[
A(x, t) = e^{-i\Lambda^2 t} \frac{\lambda}{\sqrt{2k_0^2}} \frac{\cos(k_0(x - V_{phase} t))}{\cosh(\lambda(x - C_g t))}
\]

(4)

\[
\Lambda^2 = \frac{\omega_0 \lambda^2}{8k_0^2}.
\]

Wavetrain of the amplitude \( a \) with wavenumber \( k_0 \) is unstable with respect to large scale modulation \( \delta k \). Growth rate of the instability \( \gamma \) is

\[
\gamma = \frac{\omega_0}{2} \left( \left( \frac{\delta k}{k_0} \right)^2 (ak_0)^2 - \frac{1}{4} \left( \frac{\delta k}{k_0} \right)^4 \right)^{\frac{1}{2}}.
\]

(5)

Here \( \omega_0 = \sqrt{gk_0} \).
Conformal mapping

Domain on $Z$-plane $Z = x + iy$,

$$-\infty < x < \infty, \quad -\infty < y \leq \eta(x, t),$$

to the lower half-plane,

$$-\infty < u < \infty, \quad -\infty < v \leq 0,$$

$W$-plane

\[ W = u + iv. \]
Equations for $Z$ and $\Phi$

If conformal mapping has been applied then it is naturally introduce complex analytic functions

$$Z = x + iy, \quad \text{and complex velocity potential} \quad \Phi = \Psi + i\hat{H}\Psi.$$  

$$Z_t = iUZ_u,$$

$$\Phi_t = iU\Phi_u - \hat{P}\left(\frac{|\Phi_u|^2}{|Z_u|^2}\right) + ig(Z - u).$$

$U$ is a complex transport velocity:

$$U = \hat{P}\left(\frac{-\hat{H}\Psi_u}{|Z_u|^2}\right). \quad u \rightarrow w$$

Projector operator $\hat{P}(f) = \frac{1}{2}(1 + i\hat{H})(f)$. 

About Shape of Freakon – p. 6
Cubic equations for $R$ and $V$

Surface dynamics (and the fluid bulk!) is described by two analytic functions, $R(w,t)$ and $V(w,t)$. They are related to conformal mapping $Z$ and complex velocity potential:

$$R = \frac{1}{Z_w}, \quad \Phi_w = -iVZ_w.$$ 

For $R$ and $V$ dynamic equations have the simplest form:

$$R_t = i[UR' - U'R],$$

$$V_t = i[UV' - B'R] + g(R - 1).$$

Complex transport velocity $U$ is defined as

$$U = \hat{P}(V\bar{R} + \bar{V}R), \quad \text{and} \quad B = \hat{P}(V\bar{V}).$$
NLSE and Dysthe and Conformal variables - I

Consider weakly nonlinear wave train. Use $r$ instead of $R$

$$r = R - 1.$$  

Then equations for $R$ and $V$ transform into

$$r_t + iV' = i(-U' + Vr' - V'r + Ur' - rU'),$$

$$V_t - gr = i(VV' - B' + UV' - rB').$$  \hspace{1cm} (6)$$

$$U = \hat{P}(V\bar{r} + \bar{V}r).$$  \hspace{1cm} (7)$$
NLSE and Dysthe and Conformal variables - II

We will look for the breather solution. It is periodic in some reference frame moving with velocity $c$. In this reference frame equations for $r$ and $V$ read

$$r_t - cr' + iV' = i(-U' + Vr' - V'r + Ur' - rU') = F,$$
$$V_t - cV' - gr = i(VV' - B' + UV' - rB') = G.$$  

We look for the solution of these equations in the following form

$$r = \sum_{n=0}^{\infty} r_n(u, t)e^{in(\Omega t - ku)}, \quad k > 0$$

$$V = \sum_{n=0}^{\infty} V_n(u, t)e^{in(\Omega t - ku)}. \quad (8)$$
Thereafter we will put \( k = 1, \ c = \frac{1}{2}, \ \Omega = \frac{1}{2} \). The leading terms in expansion (8) are

\[
\begin{align*}
    r_1, \\
    V_1 &\sim \epsilon << 1.
\end{align*}
\]

Then

\[
\begin{align*}
    r_n &\sim V_n \sim \epsilon^n, \\
    r_0 &\sim V_0 \sim \epsilon^3.
\end{align*}
\]

(9)

\( r_n, V_n \) are "slow" functions of \( u \). In other words

\[
\begin{align*}
    \frac{r'_n}{r_n} &\sim \frac{V'_n}{V_n} \sim \epsilon << 1.
\end{align*}
\]

(10)

For the slow componets (time derivatives)

\[
\begin{align*}
    \frac{\dot{r}_n}{r_n} &\sim \frac{\dot{V}_n}{V_n} \sim \epsilon^2 << 1.
\end{align*}
\]

(11)
NLSE and Dysthe and Conformal variables - IV

To proceed in derivation of envelope equation we have to learn how to calculate projective operator of functions like \( a(u)e^{imu} \). Here \( a(u) \) - any "slow" function of \( u \).

\[
\hat{P}(e^{ikm_a(u)}) = \begin{cases} 
0, & m > 0, \\
\epsilon^{ikm_a(u)}, & m < 0 
\end{cases} 
\] (12)

Only if \( m = 0 \), projection is a nontrivial operation. Thereafter we put

\[ V_1 = \epsilon \psi \]

and replace

\[
\frac{\partial}{\partial u} \rightarrow \epsilon \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial t} \rightarrow \epsilon^2 \frac{\partial}{\partial t}.
\] (13)
NLSE and Dysthe and Conformal variables - V

Using the rule (12) we find with accuracy up to $\epsilon^3$

$$V_2 = \epsilon^2(-i\psi^2 + \frac{\epsilon}{2}\psi\psi'),$$
$$r_2 = \epsilon^2(\psi^2 + i\epsilon\psi\psi'),$$
$$r_0 = i\epsilon^3\hat{P}(|\psi|^2),\quad V_0 = \epsilon^2\hat{P}(|\psi|^2)' \quad (14)$$

$r_1$ and $V_1$ are related with relation

$$r_1 = V_1 - \frac{\epsilon}{2}V_1' \quad (15)$$

$$2i\dot{\psi} + \frac{1}{4}\psi'' + |\psi|^2\psi = \epsilon \left[ \dot{\psi}' - \psi \hat{H}(|\psi|^2)' - 2i(|\psi|^2\psi)' \right] \quad (16)$$

This is the Dysthe equation in conformal variables. In the limit of $\epsilon \to 0$ it gives standard NLSE.
Stationary Solution - FREAKON

$$\psi = A(u)e^{i\Phi}e^{\frac{it}{2}}.$$ 

$A(u)$ and $\Phi$ - are real functions satisfying the equations

$$-A + \frac{1}{4}A'' + A^3 - \frac{1}{4}A\Phi'^2 = -\epsilon \left\{ \left( \frac{1}{2} + 2A^2 \right)\Phi' + A\hat{K}A^2 \right\}. \quad (17)$$

$$\Phi' = \epsilon(1 - 6A^2). \quad (18)$$

Keeping in (17) terms of the order of $\epsilon^2$ is exceeding of accuracy. Thus it can be simplified up to the form

$$-A + \frac{1}{4}A'' + A^3 + \epsilon A\hat{K}A^2 = 0. \quad (19)$$

$\hat{K}$ is pure negative, $\hat{K}e^{iku} = -|k|e^{iku}$. 
Stationary Solution - FREAKON

Equation (19) realize minimum of the functional

\[ H = \int_{-\infty}^{\infty} \left\{ -\frac{1}{2}A^2 - \frac{1}{8}A'^2 + \frac{1}{4}A^4 + \frac{\epsilon}{4}A^2 \hat{K}A^2 \right\} , \frac{\partial H}{\partial A} = 0. \quad (20) \]

Let us \( A = \frac{a}{\cosh 2u} \). \( a \) - is still unknown value. As a result

\[ H = -\frac{2}{3}a^2 + \left( \frac{1}{6} - 0.22\epsilon \right)a^4. \]

Condition \( \frac{\partial H}{\partial A} = 0 \) gives

\[ a = \sqrt{\frac{2}{1 - 1.32\epsilon}}. \quad (21) \]

In the limit \( \epsilon \to 0 \) we get the NLSE result, \( a = \sqrt{2} \). One can see that relatively small \( \epsilon \) leads to the strong deviation from the NLSE limit.
We compare breather-type solution with the soliton shape. In the Figure envelope is the following:

\[ A = \frac{a}{\cosh \lambda x}. \]

with \( a = 0.0084 \), and \( \lambda = 17 \). If it were NLSE envelope with the same \( \lambda = 17 \), than \( a \) would be 0.0048.
NLSE solitons are lower and wider. This is in agreement with the theory.

\[
\lambda = 4.0, \quad \epsilon = 0.290.
\]

Figure 2. Solitons for NLSE and Dysthe equation.
Figure 3. Solitons for NLSE and Dysthe equation. 
\( \lambda = 15.0, \epsilon = 0.070. \)
Giant Breather, $k$-$\omega$ spectrum

Figure 4. Negative frequency is absent!