Cluster Mutation-Periodic Quivers and Associated Laurent Sequences

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Outline

- Background on the Somos 4 Sequence
- Quivers and Mutation
- Periodicity and Primitives
- Cluster Variables
- Maps of the Line and Plane
- The Question of Integrability
- Supersymmetric Quiver Gauge Theories
- Conclusions, Questions and Perspectives

Based on the paper: Allan P. Fordy and Robert J. Marsh,

arXiv:0904.0200v2 [math.CO].

Somos 4 Sequence and the Laurent Property

Iteration on the real line

$$x_n x_{n+4} = x_{n+1} x_{n+3} + x_{n+2}^2.$$

Specify x_1, x_2, x_3, x_4 and calculate.

For example, $1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209 \ldots$

Already at $n = 5$ we need to divide by 2 to calculate $x_9 = 59$.

And why should $\dfrac{59 \times 1529 + 314^2}{23}$ be an **integer**??

Every term in the sequence is integer.

Arbitrary initial conditions:

$$s, t, u, v, \frac{u^2 + tv}{s}, \frac{u^3 + tuv + sv^2}{st}, \ldots$$

$$x_9 = \frac{4\,t^5\,u^2\,v^3 + t^6\,v^4 + \cdots + t^3\,\left(4\,u^6\,v + 7\,s\,u^3\,v^3 + 2\,s^2\,v^5\right)}{s^3\,t^3\,u^2\,v}.$$

Why should $x_9 = \dfrac{x_6 x_8 + x_7^2}{x_5}$ **not** have $u^2 + tv$ in the denominator??

Each term in the sequence is a Laurent polynomial.

ie Each term is **polynomial** in $s, t, u, v, s^{-1}, t^{-1}, u^{-1}, v^{-1}$.

The Somos N iteration

This is

$$x_n x_{n+N} = \sum_{r=1}^{[N/2]} x_{n+r} x_{n+N-r}$$

and has the Laurent property for $N = 4, 5, 6, 7$.

These are also Poisson maps with first integrals and even **integrable**.

Somos 4 is related to the symmetric QRT map.

Maps with the Laurent property are not generally integrable.

Integrable maps do not generally have the Laurent property.

But plenty have both!

Cluster Algebras and the Laurent phenomenon.

The Laurent phenomenon occurs in many contexts.

S Fomin and A Zelevinsky developed a general theory in

- *The Laurent phenomenon.* Advances in Applied Mathematics, 28:119–144, 2002.
- *Cluster algebras I: Foundations.* J Amer Math Soc, 15:497–529, 2002.

We exploit the cluster connection.

We require the related **quiver** to have a periodicity property which implies the cluster relations are of **iterative type**.

We classify such quivers and such iterations.

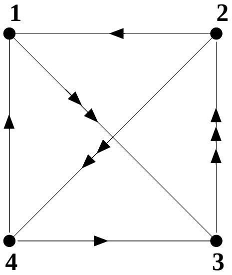
What is a Quiver?

A quiver is a type of **directed graph** N nodes with directed edges between them.

There may be several arrows between a given pair of vertices.

For cluster algebras there should be no 1 cycles or 2-cycles.

Such quivers can be represented by a skew-symmetric matrix B, with $B_{ij} = r$ if there are r arrows going **from** node i **to** node j.



$$B = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix}$$

Figure: The Somos 4 quiver.

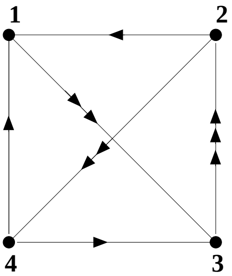
Quiver Mutation (Fomin and Zelevinsky)

The mutation of Q at node k, denoted by $\mu_k Q$, is constructed (from Q) as follows:

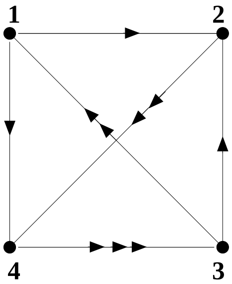
1. Reverse all arrows which either originate or terminate at node k.
2. Suppose that there are p arrows from node i to node k and q arrows from node k to node j (in Q). Add pq arrows going from node i to node j to any arrows already there.
3. Remove (both arrows of) any two-cycles created in the previous steps.

In Step 2, pq is just the number of paths of length 2 between nodes i and j which pass through node k.

The Somos 4 Quiver (Period 1 mutation)



(a) The quiver, S_4.



(b) Mutation of S_4 at 1.

This is a complicated combinatorial operation, which remarkably (for this quiver) leads to a simple rotation of the diagram.

Mutation at **node 2** of this second quiver results in a further rotation.

Mutation at **node 3** of this third quiver.......

This is an example of a **period** 1 **quiver**.

Periodicity $\Rightarrow$ **repetition of formula** ($=$ **iteration).**

Matrix Mutation

Whilst the quiver gives a nice pictorial representation, calculations are carried out using the matrix representation.

Let B and $\tilde{B}$ be the skew-symmetric matrices corresponding to the quivers Q and $\tilde{Q} = \mu_k Q$.

Let b_{ij} and $\tilde{b}_{ij}$ be the corresponding matrix entries. Then quiver mutation amounts to the following formula

$$\tilde{b}_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \textit{otherwise}. \end{cases}$$

Definition of Periodicity

Consider a sequence of mutations:

Mutation at node 1 of a quiver $Q(1)$ gives a second quiver $Q(2)$.

Mutation at node 2 of a quiver $Q(2)$ gives a third quiver $Q(3)$, etc.

A quiver Q has *period m* if it satisfies

$$Q(m+1) = \rho^m Q(1),$$

with the mutation sequence depicted by

$$Q = Q(1) \xrightarrow{\mu_1} Q(2) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{m-1}} Q(m) \xrightarrow{\mu_m} Q(m+1) = \rho^m Q(1).$$

where the permutation $\rho : (1, 2, \cdots, n) \to (n, 1, \cdots, n-1)$.

The the above sequence of quivers is called the *periodic chain* associated to Q.

Matrix Version of Periodicity

Let $B(i)$ be the matrix representing quiver $Q(i)$.

$$B(1) \xrightarrow{\mu_1} B(2) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{m-1}} B(m) \xrightarrow{\mu_m} B(m+1) = \rho^m B(1) \rho^{-m}.$$

where the permutation $\rho : (1, 2, \cdots, n) \to (n, 1, \cdots, n-1)$, is represented by the matrix

$$\rho = \begin{pmatrix} 0 & \cdots & \cdots & 1 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}.$$

The Somos 4 quiver is period 1.

Period 1 Primitives

The first task is to classify all periodic quivers.

To date, we have a full classification of period 1 quivers,
a partial classification of period 2 quivers
and examples of higher period ones.

The general period 1 quiver is built out of primitives.

A period 1 primitive is a period 1 quiver for which node 1 is a sink.

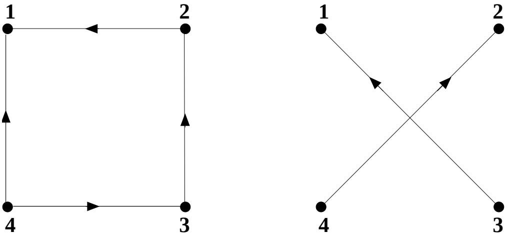


Figure: The period 1 primitives for 4 nodes.

Period 1 Primitives. 5 and 6 Nodes

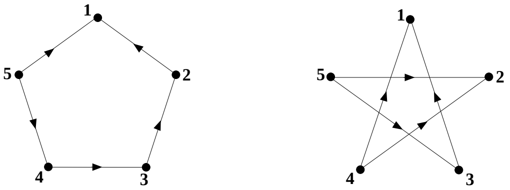


Figure: The period 1 primitives for 5 nodes.

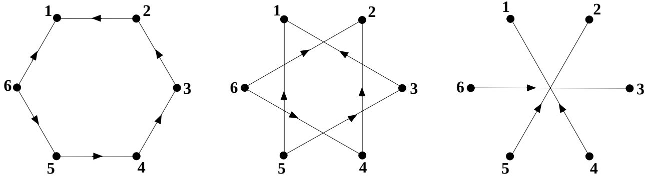
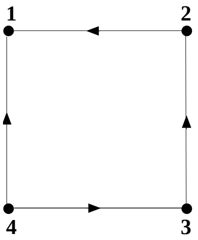
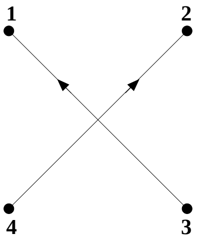


Figure: The period 1 primitives for 6 nodes.

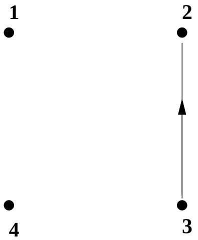
Somos 4 Quiver



(a) 1 of these

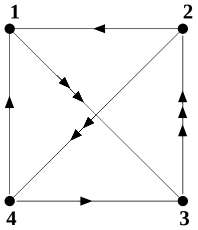


(b) -2 of these



(c) $+2$ of these

gives



Period 1 Quiver with 6 Nodes

With $\delta_{ij} = \frac{1}{2}(m_i|m_j| - m_j|m_i|)$. $(= 0$ if m_i, m_j same sign.)

$$B = \begin{pmatrix} 0 & -m_1 & -m_2 & -m_3 & -m_2 & -m_1 \\ m_1 & 0 & -m_1 & -m_2 & -m_3 & -m_2 \\ m_2 & m_1 & 0 & -m_1 & -m_2 & -m_3 \\ m_3 & m_2 & m_1 & 0 & -m_1 & -m_2 \\ m_2 & m_3 & m_2 & m_1 & 0 & -m_1 \\ m_1 & m_2 & m_3 & m_2 & m_1 & 0 \end{pmatrix}$$

$$+ \left(\begin{array}{c|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\delta_{12} & -\delta_{13} & -\delta_{12} & 0 \\ 0 & \delta_{12} & 0 & -\delta_{12} & -\delta_{13} & 0 \\ 0 & \delta_{13} & \delta_{12} & 0 & -\delta_{12} & 0 \\ 0 & \delta_{12} & \delta_{13} & \delta_{12} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\delta_{23} & 0 & 0 \\ 0 & 0 & \delta_{23} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

6 node quiver $+$ 4 node quiver $+$ 2 node quiver.

Cluster Variables

Attach a variable at each node, labelled $(x_1, \cdots, x_n)$.

Quiver mutation: change the associated matrix and transform the cluster variables

$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, \tilde{x}_k, \cdots, x_n),$$

with the (cluster) **exchange relation**.

$$x_k \tilde{x}_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}, \qquad \tilde{x}_i = x_i \text{ for } i \neq k.$$

If one of these products is empty (which occurs when all b_{ik} have the same sign) then it is replaced by the number 1.

Just depends upon the k^{th} column of the matrix.

Since the matrix is skew-symmetric, variable x_k **does not** occur on the right side of the formula.

Period 1 Quivers and Maps of the Line

Cluster variables $(x_1, \cdots, x_n)$, with x_k situated at node k.

Successively mutate at nodes $1, 2, 3, \ldots$ and define

$$x_{n+1} = \tilde{x}_1, \quad x_{n+2} = \tilde{x}_2, \quad etc.$$

The **exchange relation** gives us a formula of the type

$$x_k x_{k+n} = F(x_{k+1}, \cdots, x_{k+n-1}),$$

with F being the sum of two monomials.

n^{th} order map of the real line, with initial conditions

$$x_i = c_i \quad \text{for} \quad i = 1, \cdots, n.$$

However, F is derived through the cluster exchange relation, so x_k is a **Laurent polynomial** in $c_1, \cdots, c_n$, for all k. (Fomin and Zelevinsky)

In particular, starting with $c_i = 1, i = 1, \cdots n$, then x_k is **integer** for all k.

4 Node Case

Let $m_1 = r, m_2 = -s$, both r and s positive. Then

$$B(1) = \begin{pmatrix} 0 & -r & s & -r \\ r & 0 & -r(1+s) & s \\ -s & r(1+s) & 0 & -r \\ r & -s & r & 0 \end{pmatrix}$$

Mutate at node 1, with $(x_1, x_2, x_3, x_4) \mapsto (x_5, x_2, x_3, x_4)$.

Cluster exchange relation $x_1 x_5 = x_2^r x_4^r + x_3^s$,

Mutate $Q(2)$ **at node 2**, with $(x_5, x_2, x_3, x_4) \mapsto (x_5, x_6, x_3, x_4)$,

$$x_2 x_6 = x_3^r x_5^r + x_4^s,$$

Just a shift of the above, since $\mu_1 B(1) = \rho B(1) \rho^{-1}$.

Generally $x_n x_{n+4} = x_{n+1}^r x_{n+3}^r + x_{n+2}^s$.

When $r = 1, s = 2$, this is exactly the Somos 4 sequence.

5 and 6 Node Cases

For $n = 5$, we get

$$x_n x_{n+5} = x_{n+1}^r x_{n+4}^r + x_{n+2}^s x_{n+3}^s,$$

which reduces to Somos 5 when $r = s = 1$.

For $n = 6$, there are 3 parameters m_i. Different equations depending upon relative signs.

Cannot get Somos 6, which has 3 terms:

$$x_n x_{n+6} = x_{n+1} x_{n+5} + x_{n+2} x_{n+4} + x_{n+3}^2$$

The case $m_1 = r, m_2 = -s, m_3 = 0$ **with** r, s **positive:**

$$x_n x_{n+6} = x_{n+1}^r x_{n+5}^r + x_{n+2}^s x_{n+4}^s,$$

which gives the first two terms of Somos 6 when $r = s = 1$.

The case $m_1 = r, m_2 = 0, m_3 = -s$ **with** r, s **positive:**

$$x_n x_{n+6} = x_{n+1}^r x_{n+5}^r + x_{n+3}^s.$$

Period 2 Quivers and Maps of the Plane

Cluster variables $(z_1, \cdots, z_n)$, with z_k situated at node k.

Successively mutate at nodes $1, 2, 3, \dots$ and define

$$z_{n+1} = \tilde{z}_1, \quad z_{n+2} = \tilde{z}_2, \ \ etc.$$

However, exchange relation now gives **alternating pair** of formulae

$$\begin{aligned} z_{2k-1} z_{2k-1+n} &= F_0(z_{2k}, \cdots, z_{2k+n-2}), \\ z_{2k} z_{2k+n} &= F_1(z_{2k+1}, \cdots, z_{2k+n-1}), \end{aligned}$$

with $k = 1, 2, \dots$, and F_i being the sum of two monomials.

It is natural, therefore, to relabel the cluster variables as

$$x_k = z_{2k-1}, \quad y_k = z_{2k}$$

and to interpret the map as acting on the $x - y$ plane.

4 and 5 Node Cases

The 4 node case gives a second order map of the plane:

$$x_n x_{n+2} = y_n^r y_{n+1}^t + x_{n+1}^s, \quad y_n y_{n+2} = x_{n+1}^t x_{n+2}^r + y_{n+1}^s.$$

When $t = r$ this reduces to period 1.

The 5 node case gives a third order map of the plane:

$$\begin{aligned} y_n x_{n+3} &= y_{n+2}^r x_{n+1}^t + x_{n+2} y_{n+1}, \\ x_{n+1} y_{n+3} &= y_{n+1}^r x_{n+3}^t + x_{n+2} y_{n+2}, \end{aligned}$$

for $n = 1, 2, \ldots,$ together with

$$x_1 y_3 = y_1^r x_3^t + x_2 y_2,$$

and initial conditions $(x_1, y_1, x_2, y_2, x_3) = (c_1, c_2, c_3, c_4, c_5)$.

The Question of Integrability

Whilst these maps are guaranteed to have the Laurent property, they will usually fail to be integrable (positive algebraic entropy (Claud Viallet)). For instance,

$$x_n x_{n+4} = x_{n+1}^r x_{n+3}^r + x_{n+2}^s, \quad r, s \geq 0,$$

is integrable for $r = 1, s = 0, 1, 2$, but otherwise not (Hone).

Maps from our primitives are linearisable.

The primitive $P_N^{(1)}$ has, tt node 1, 2 **incoming** arrows, but none outgoing, giving the N^{th} order map

$$x_n x_{n+N} = x_{n+1} x_{n+N-1} + 1.$$

The elements of this sequence also satisfy the linear map

$$x_n + x_{n+2(N-1)} = S_{N,1} x_{n+N-1},$$

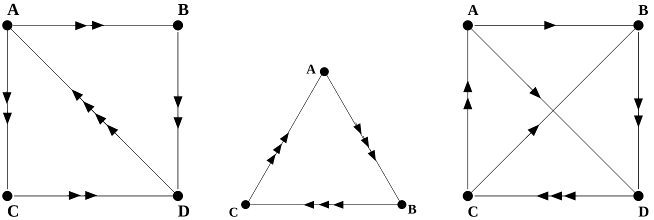
where $S_{N,1}$ is one of the first integrals of the nonlinear map.

Supersymmetric Quiver Gauge Theories

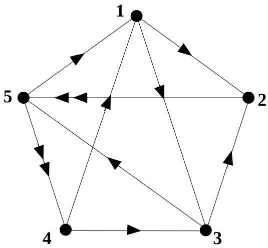
Curiously, our quivers arise in the context of *quiver gauge theories* in the $D-$*brane* literature.

The Seiberg dual theory has a new quiver, obtained using a combinatorial rule from the original quiver using the choice of vertex.

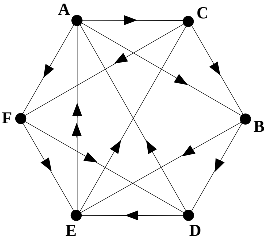
The combinatorial rule for Seiberg-dualising a quiver coincides with the rule for Fomin-Zelevinsky quiver mutation.



(d) Hirzebruch 0 (e) del Pezzo 0 (f) del Pezzo 1



(g) del Pezzo 2



(h) del Pezzo 3

After relabelling, these 5 quivers belong to our classification. There are respectively period 2, period 1, period 1, period 1 and period 2.

After our paper first appeared in the arXiv, Jan Stienstra pointed out to us that these 5 quivers also appear in his paper in the context of Gelfand-Kapranov-Zelevinsky hypergeometric systems in two variables, suggesting a possible connection between cluster mutation and such systems.

Conclusions, Questions and Perspectives

- Cluster relations of mutation periodic quivers give iterations with the Laurent property. Examples presented here had fixed coefficients. Can extend the action of the rotation to “frozen vertices” and include parameter coefficients in the maps.
- A limitation is that we can only have 2 terms on the “right hand side”. Somos 6 (for example) has 3 terms. Is it possible to generalise the cluster construction in some way?
- Classification of integrable and/or linearisable cases?
- An understanding of why quivers which appear in the Quiver Gauge Theory literature are special cases of our periodic quivers.