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SOLITONS, COLLAPSES AND TURBULENCE: VLADIMIR ZAKHAROV’s 70 BIRTHDAY
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**KP equation**

\[
(u_t - uu_x - u_{xxx})_x = u_{yy}
\]

Perturbative symmetry approach

\[
(u_t - \varepsilon uu_x - u_{xxx})_x = u_{yy}
\]

Dispersive deformation

\[
(u_t - uu_x - \varepsilon^2 u_{xxx})_x = u_{yy}
\]

**Program of classification of (2+1)D integrable systems:**

- Classify (2+1)D dispersionless systems which may (potentially) arise as dispersionless limits of integrable soliton equations (method of hydrodynamic reductions)
- Understand how to add dispersive corrections (deformation of hydrodynamic reductions)
Plan:

- The method of hydrodynamic reductions. Example of dKP
- Classification of (2+1)D dispersionless integrable systems
  - Systems of hydrodynamic type
  - Hydrodynamic chains
  - Equations of the dispersionless Hirota type
  - Second order quasilinear PDEs
- Dispersive deformations of dispersionless integrable systems
- Classification of third order (2+1)D soliton equations with ‘simplest’ nonlocalities
The method of hydrodynamic reductions

Applies to quasilinear equations

\[ A(u)u_x + B(u)u_y + C(u)u_t = 0 \]

Consists of seeking N-phase solutions

\[ u = u(R^1, ..., R^N) \]

The phases \( R^i(x, y, t) \) are required to satisfy a pair of commuting equations

\[ R_{xy}^i = \mu^i(R) R_x^i, \quad R_t^i = \lambda^i(R) R_x^i \]

Commutativity conditions:

\[ \frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \]

**Definition**

A quasilinear system is said to be integrable if, for any number of phases N, it possesses infinitely many N-phase solutions parametrized by 2N arbitrary functions of one variable.
Example of dKP

\[(u_t - uu_x)_x = u_{yy}\]

First order (hydrodynamic) form:

\[u_t - uu_x = w_y, \quad u_y = w_x\]

\[N\text{-phase solutions: } u = u(R^1, ..., R^N), \quad w = w(R^1, ..., R^N)\text{ where}\]

\[R^i_y = \mu^i(R)R^i_x, \quad R^i_t = \lambda^i(R)R^i_x\]

Then

\[\partial_i w = \mu^i \partial_i u, \quad \lambda^i = u + (\mu^i)^2\]

Equations for \(u(R)\) and \(\mu^i(R)\) (Gibbons-Tsarev system):

\[\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \quad \partial_i \partial_j u = 2 \frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}\]

**In involution!** General solution depends on \(N\) arbitrary functions of one variable.
Generalized dKP

\[(u_t - f(u)u_x)_x = u_{yy}\]

First order (hydrodynamic) form:

\[u_t - f(u)u_x = w_y, \quad u_y = w_x\]

\(N\)-phase solutions: \(u = u(R^1, \ldots, R^N), \ w = w(R^1, \ldots, R^N)\) where

\[R_{iy}^i = \mu^i(R)R_{x}^i, \quad R_{t}^i = \lambda^i(R)R_{x}^i\]

Then

\[\partial_i w = \mu^i \partial_i u, \quad \lambda^i = f(u) + (\mu^i)^2\]

Equations for \(u(R)\) and \(\mu^i(R)\) (generalized Gibbons-Tsarev system):

\[\partial_j \mu^i = f'(u) \frac{\partial_j u}{\mu^j - \mu^i}, \quad \partial_i \partial_j u = 2f'(u) \frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}\]

Involutivity \(\rightarrow f''' = 0\)
Systems of hydrodynamic type in (2+1)D

\[ u_t + A(u)u_x + B(u)u_y = 0 \]


Nijenhuis tensor

\[ N_{jk}^i = V_j^p \partial_u^p V_k^i - V_k^p \partial_u^p V_j^i - V_p^i (\partial_u^j V_k^p - \partial_u^k V_j^p) \]

Haantjes tensor

\[ H_{jk}^i = N_{pr}^i V_j^p V_k^r - N_{jr}^p V_k^p V_j^r - N_{rk}^p V_j^r V_k^p + N_{jk}^p V_r^i V_p^r \]

General case: Integrability \implies H(V) = 0 where \( V = (A + kE)^{-1}(B + lE) \).

Hamiltonian case: Integrability \iff H(V) = 0.
Hydrodynamic chains

\[ u_t + V(u)u_x = 0 \]


Haantjes tensor \( H(V) \) well-defined!

Conservative chains

\[ u_1^t = f(u^1, u^2)_x, \quad u_2^t = g(u^1, u^2, u^3)_x, \quad u_3^t = h(u^1, u^2, u^3, u^4)_x, \ldots \]

Hamiltonian chains

\[ u_t = \left( B \frac{d}{dx} + \frac{d}{dx} B^t \right) \frac{\partial h}{\partial u}, \quad h(u^1, u^2, u^3) \]

Generic case: \( h = (u^3 + P(u^1, u^2))^{1/3} \) where \( P \) is a cubic polynomial.
Equations of the dispersionless Hirota type

\[ F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0 \]


\[ e^{u_{xx}} + e^{u_{yy}} = e^{u_{tt}} \]

\[ u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6} \eta(u_{xx})u_{xt}^2 \]

where \( \eta \) solves the Chazy equation.

21-dimensional moduli space, equivalence group \( Sp(6, R) \)

Geometry: hypersurfaces in the Lagrangian Grassmannian
Second order quasilinear PDEs

\[ f_{11}u_{xx} + f_{22}u_{yy} + f_{33}u_{tt} + 2f_{12}u_{xy} + 2f_{13}u_{xt} + 2f_{23}u_{yt} = 0 \]

\( f_{ij} \) depend on the first order derivatives \( u_x, u_y, u_t \) only.


\[ u_{xx} + u_{yy} - e^{u_t}u_{tt} = 0 \]

\[ \alpha \frac{\varphi'(u_x) - \varphi'(u_y)}{\varphi(u_x)\varphi(u_y)} u_{xy} + \beta \frac{\varphi'(u_t) - \varphi'(u_x)}{\varphi(u_x)\varphi(u_t)} u_{xt} + \gamma \frac{\varphi'(u_y) - \varphi'(u_t)}{\varphi(u_y)\varphi(u_t)} u_{yt} = 0 \]

20-dimensional moduli space, equivalence group \( SL(4, R) \)

Geometry: conformal structures in projective space
Dispersive deformations of dispersionless integrable systems


\[
(u_t - uu_x - \varepsilon^2 u_{xxx})_x = u_{yy}
\]

Look for deformed N-phase solutions in the form

\[
u = u(R^1, ..., R^N) + \varepsilon^2(\ldots) + \varepsilon^4(\ldots) + \ldots
\]

where

\[
R^i_y = \mu^i(R) R^i_x + \varepsilon^2(\ldots) + \varepsilon^4(\ldots) + \ldots
\]

\[
R^i_t = \lambda^i(R) R^i_x + \varepsilon^2(\ldots) + \varepsilon^4(\ldots) + \ldots
\]

Here (\ldots) are required to be polynomial and homogeneous in the derivatives of \( R^i \). Recall that \( \lambda^i = u + (\mu^i)^2 \), and \( \mu^i, u \) satisfy the Gibbons-Tsarev system.
Deformations of one-phase reductions of dKP

\[ (u_t - uu_x - \epsilon^2 u_{xxx})_x = u_{yy} \]

Deformed one-phase reductions (modulo the Miura group can assume \( u = R \)):

\[ R_y = \mu R_x \]
\[ + \epsilon^2 \left( \mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3)R_x^2 \right)_x + O(\epsilon^4) \]

\[ R_t = (\mu^2 + R)R_x \]
\[ + \epsilon^2 \left( (2\mu\mu' + 1)R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2)R_x^2 \right)_x + O(\epsilon^4) \]

Conjecture

For any soliton system in (2+1)D, all hydrodynamic reductions of its dispersionless limit can be deformed into reductions of the dispersive counterpart (linear non-degeneracy of the dispersionless limit is required).
Generalized KP equation

\[ u_t - uu_x + \varepsilon (A_1 u_{xx} + A_2 u_x^2) + \varepsilon^2 (B_1 u_{xxx} + B_2 u_x u_{xx} + B_3 u_x^3) = w_y \]

\[ w_x = u_y \]

Require that all one-phase reductions can be deformed as

\[ u = R, \quad w = w(R) + \varepsilon^2 (\ldots) + \varepsilon^4 (\ldots) + \ldots \]

where

\[ R_y = \mu R_x + \varepsilon^2 (\ldots) + \varepsilon^4 (\ldots) + \ldots \]

\[ R_t = (\mu^2 + R) R_x + \varepsilon^2 (\ldots) + \varepsilon^4 (\ldots) + \ldots \]

\[ w' = \mu. \text{ This gives } A_1 = A_2 = B_2 = B_3 = 0, \quad B_1=\text{const}, \quad \implies \quad \text{KP} \]
Classification result: scalar third order (2+1)D soliton equations with simplest nonlocalities


\[ u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\ldots) + \epsilon^2(\ldots), \quad w_x = u_y \]

here \( \varphi, \psi, \eta \) are functions of \( u \) and \( w \), and \((\ldots)\) denote terms which are polynomial in the derivatives of \( u \) and \( w \) with respect to \( x \) and \( y \) of orders 2 and 3, respectively. Here \( w = D_x^{-1}D_y u \) is the nonlocality, no other non-local variables are allowed.

- Classify integrable dispersionless systems of the form

\[ u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y \]

- Add dispersive corrections which inherit all hydrodynamic reductions (sufficient to consider 1-component reductions only)
Known examples

**KP**

\[ u_t = uu_x + wy + \epsilon^2 u_{xxx} \]

**mKP**

\[ u_t = (w - \frac{u^2}{2})u_x + wy + \epsilon^2 u_{xxx} \]

**Gardner**

\[ u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + wy + \epsilon^2 u_{xxx} \]

**VN**

\[ u_t = (uw)_y + \epsilon^2 u_{yyy} \]

**mVN**

\[ u_t = (uw)_y + \epsilon^2 \left( u_{yy} - \frac{3}{4} \frac{u_y^2}{u} \right)_y \]

**HarryDym**

\[ u_t = -2wu_y + uw_y \frac{\epsilon^2}{u} \left( \frac{1}{u} \right)_{xxx} \]
Classification of integrable dispersionless limits


\[
\begin{align*}
\phi_{uu} &= -\frac{\phi_w^2 + \psi_u \phi_w - 2\psi_w \phi_u}{\eta}, \\
\phi_{uw} &= \frac{\eta_w \phi_u}{\eta}, \\
\phi_{ww} &= \frac{\eta_w \phi_w}{\eta}, \\
\psi_{uu} &= -\frac{\psi_w \phi_w + \psi_u \psi_w - 2\phi_w \eta_u + 2\eta_w \phi_u}{\eta}, \\
\psi_{uw} &= \frac{\eta_w \psi_u}{\eta}, \\
\psi_{ww} &= \frac{\eta_w \psi_w}{\eta}, \\
\eta_{uu} &= -\frac{\eta_w (\phi_w - \psi_u)}{\eta}, \\
\eta_{uw} &= \frac{\eta_w \eta_u}{\eta}, \\
\eta_{ww} &= \frac{\eta_w^2}{\eta},
\end{align*}
\]

In involution, straightforward to solve: \( \eta = 1, \eta = u, \eta = e^w h(u) \)

Conjecture

For any \( \phi, \psi, \eta \) one can reconstruct (non-uniquely) dispersive corrections which inherit all hydrodynamic reductions. Infinite series in \( \epsilon \) are required in general.
New examples: 1

\[ u_t = (\beta w + \beta^2 u^2)u_x - 3\beta uu_y + w_y + \epsilon^2[B^3(u) - \beta B^2(u)u_x], \quad w_x = w_y, \]

\[ B = \beta uD_x - D_y. \]

Lax pair

\[ \psi_{xy} = \beta uu\psi_{xx} + \frac{1}{3\epsilon^2}\psi, \]

\[ \psi_t = \beta^3 \epsilon^2 u^3 \psi_{xxx} - \epsilon^2 \psi_{yyy} + 3\beta^2 \epsilon^2 uu_y \psi_{xx} + \beta w \psi_x. \]
New examples: 2

\[ u_t = \frac{4}{27} \gamma^2 u^3 u_x + (w + \gamma u^2) u_y + uw_y + \varepsilon^2 [B^3(u) - \frac{1}{3} \gamma u_x B^2(u)], \quad w_x = u_y, \]

\[ B = \frac{1}{3} \gamma u D_x + D_y. \]

Lax pair

\[ \psi_{xy} = -\frac{\gamma}{3} u \psi_{xx} - \frac{1}{3\varepsilon^2} u \psi, \]

\[ \psi_t = \frac{\varepsilon^2 \gamma^3}{27} u^3 \psi_{xxx} + \varepsilon^2 \psi_{yyy} - \frac{\varepsilon^2 \gamma^2}{3} uu_y \psi_{xx} + \frac{\gamma^2}{27} u^3 \psi_x + w \psi_y - \frac{\gamma}{3} uu_y \psi \]
New examples: 3

\[ u_t = \frac{\delta}{u^3} u_x - 2wu_y + uw_y \frac{\epsilon^2}{u} \left( \frac{1}{u} \right)_{xxx}, \quad w_x = u_y, \]

\( \delta = 0 \) gives the Harry Dym equation.

Lax pair \( L_t = [A, L] \),

\[ L = \frac{\epsilon^2}{u^2} D_x^2 + \frac{\epsilon}{\sqrt{3}} D_y + \frac{\delta^2}{4u^2}, \]

\[ A = \frac{4\epsilon^2}{u^3} D_x^3 + \left( -\frac{6\epsilon^2 u_x}{u^4} + \frac{2\sqrt{3}\epsilon w}{u^2} \right) D_x^2 + \frac{\delta}{u^3} D_x + \left( -\frac{3\delta u_x}{2u^4} + \frac{\sqrt{3}\delta w}{2\epsilon u^2} \right) \]
Comparison of 1+1 and 2+1 deformation schemes

1+1 case:
\[ u_t = A(u)u_x + \varepsilon^2(\ldots) + \ldots \]
Dispersionless integrable systems form an infinite dimensional moduli space;
Terms at $\varepsilon^2$ contain extra functional freedom (central invariants).

2+1 case:
\[ u_t = A(u)u_x + B(u)u_y + \varepsilon^2(\ldots) + \ldots \]
Dispersionless integrable systems form finite dimensional moduli spaces;
Terms at $\varepsilon^2$ contain no functional freedom.

There is a hope to obtain explicit formulae in 2+1D