SOLITON EQUATIONS IN 2+1 DIMENSIONS: DEFORMATIONS OF DISPERSIONLESS LIMITS

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KP equation

$$\left(u_t - uu_x - u_{xxx}\right)_x = u_{yy}$$

Perturbative symmetry approach

$$\left(u_t - \varepsilon uu_x - u_{xxx}\right)_x = u_{yy}$$

Dispersive deformation

$$\left(u_t - uu_x - \varepsilon^2 u_{xxx}\right)_x = u_{yy}$$

Program of classification of (2+1)D integrable systems:

- Classify (2+1)D dispersionless systems which may (potentially) arise as dispersionless limits of integrable soliton equations (method of hydrodynamic reductions)
- Understand how to add dispersive corrections (deformation of hydrodynamic reductions)

Plan:

- The method of hydrodynamic reductions. Example of dKP
- Classification of (2+1)D dispersionless integrable systems
  - Systems of hydrodynamic type
  - Hydrodynamic chains
  - Equations of the dispersionless Hirota type
  - Second order quasilinear PDEs
- Dispersive deformations of dispersionless integrable systems
- Classification of third order (2+1)D soliton equations with ‘simplest’ nonlocalities

The method of hydrodynamic reductions

Applies to quasilinear equations

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0$$

Consists of seeking N-phase solutions

$$\mathbf{u} = \mathbf{u}(R^1, ..., R^N)$$

The phases $R^i(x, y, t)$ are required to satisfy a pair of commuting equations

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x$$

Commutativity conditions: $\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$

Definition

A quasilinear system is said to be integrable if, for any number of phases N, it possesses infinitely many N-phase solutions parametrized by 2N arbitrary functions of one variable.

Example of dKP

$$(u_t - uu_x)_x = u_{yy}$$

First order (hydrodynamic) form:

$$u_t - uu_x = w_y, \quad u_y = w_x$$

N-phase solutions: $u = u(R^1, ..., R^N), \; w = w(R^1, ..., R^N)$ where

$$R^i_y = \mu^i(R) R^i_x, \qquad R^i_t = \lambda^i(R) R^i_x$$

Then

$$\partial_i w = \mu^i \partial_i u, \qquad \lambda^i = u + (\mu^i)^2$$

Equations for $u(R)$ and $\mu^i(R)$ (Gibbons-Tsarev system):

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \qquad \partial_i \partial_j u = 2 \frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}$$

In involution! General solution depends on N arbitrary functions of one variable.

Generalized dKP

$$(u_t - f(u)u_x)_x = u_{yy}$$

First order (hydrodynamic) form:

$$u_t - f(u)u_x = w_y, \quad u_y = w_x$$

N-phase solutions: $u = u(R^1, ..., R^N),\ w = w(R^1, ..., R^N)$ where

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x$$

Then

$$\partial_i w = \mu^i \partial_i u, \qquad \lambda^i = f(u) + (\mu^i)^2$$

Equations for $u(R)$ and $\mu^i(R)$ (generalized Gibbons-Tsarev system):

$$\partial_j \mu^i = f'(u)\frac{\partial_j u}{\mu^j - \mu^i}, \qquad \partial_i\partial_j u = 2f'(u)\frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}$$

Involutivity $\longrightarrow f'' = 0$

Systems of hydrodynamic type in (2+1)D

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = 0$$

E. V. Ferapontov and K. R. Khusnutdinova, Double waves in multi-dimensional systems of hydrodynamic type: the necessary condition for integrability, Proc. Royal Soc. A **462** *(2006) 1197-1219.*

E.V. Ferapontov, A. Moro and V.V. Sokolov, Hamiltonian systems of hydrodynamic type in 2+1 dimensions, Comm. Math. Phys. **285***, no. 1 (2009) 31–65.*

Nijenhuis tensor

$$N^i_{jk} = V^p_j \partial_{u^p} V^i_k - V^p_k \partial_{u^p} V^i_j - V^i_p (\partial_{u^j} V^p_k - \partial_{u^k} V^p_j)$$

Haantjes tensor

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p$$

General case: Integrability $\Longrightarrow$ $H(V) = 0$ where $V = (A + kE)^{-1}(B + lE)$.

Hamiltonian case: Integrability $\Longleftrightarrow H(V) = 0$.

Hydrodynamic chains

$$\mathbf{u}_t + V(\mathbf{u})\mathbf{u}_x = 0$$

E. V. Ferapontov and D. G. Marshall, Differential-geometric approach to the integrability of hydrodynamic chains: the Haantjes tensor, Math. Ann. **339***, no. 1 (2007) 61-99.*

Haantjes tensor $H(V)$ well-defined!

Conservative chains

$$u^1_t = f(u^1, u^2)_x, \quad u^2_t = g(u^1, u^2, u^3)_x, \quad u^3_t = h(u^1, u^2, u^3, u^4)_x, \ldots$$

Hamiltonian chains

$$\mathbf{u}_t = \left(B\frac{d}{dx} + \frac{d}{dx}B^t\right)\frac{\partial h}{\partial \mathbf{u}}, \qquad h(u^1, u^2, u^3)$$

Generic case: $h = (u^3 + P(u^1, u^2))^{1/3}$ where P is a cubic polynomial.

Equations of the dispersionless Hirota type

$$F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0$$

E.V. Ferapontov, L. Hadjikos and K.R. Khusnutdinova, Integrable equations of the dispersionless Hirota type and hypersurfaces in the Lagrangian Grassmannian, arXiv: 0705.1774 (2007).

$$e^{u_{xx}} + e^{u_{yy}} = e^{u_{tt}}$$

$$u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6}\eta(u_{xx})u_{xt}^2$$

where η solves the Chazy equation.

21-dimensional moduli space, equivalence group $Sp(6, R)$

Geometry: hypersurfaces in the Lagrangian Grassmannian

Second order quasilinear PDEs

$$f_{11}u_{xx} + f_{22}u_{yy} + f_{33}u_{tt} + 2f_{12}u_{xy} + 2f_{13}u_{xt} + 2f_{23}u_{yt} = 0$$

f_{ij} depend on the first order derivatives u_x, u_y, u_t only.

P. A. Burovskii, E. V. Ferapontov and S. P. Tsarev, Second order quasilinear PDEs and conformal structures in projective space; arXiv:0802.2626v1, (2008).

$$u_{xx} + u_{yy} - e^{u_t}u_{tt} = 0$$

$$\alpha\frac{\wp'(u_x) - \wp'(u_y)}{\wp(u_x)\wp(u_y)}\, u_{xy} + \beta\frac{\wp'(u_t) - \wp'(u_x)}{\wp(u_x)\wp(u_t)}\, u_{xt} + \gamma\frac{\wp'(u_y) - \wp'(u_t)}{\wp(u_y)\wp(u_t)}\, u_{yt} = 0$$

20-dimensional moduli space, equivalence group $SL(4, R)$

Geometry: conformal structures in projective space

Dispersive deformations of dispersionless integrable systems

E. V. Ferapontov and A. Moro, Dispersive deformations of hydrodynamic reductions of 2D dispersionless integrable systems, J. Phys. A: Math. Theor. 42 (2009) 035211, 15pp.

$$\left(u_t - uu_x - \varepsilon^2 u_{xxx}\right)_x = u_{yy}$$

Look for deformed N-phase solutions in the form

$$u = u(R^1, ..., R^N) + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$

where

$$R^i_y = \mu^i(R)R^i_x + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$
$$R^i_t = \lambda^i(R)R^i_x + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$

Here $(\dots)$ are required to be **polynomial** and **homogeneous** in the derivatives of R^i. Recall that $\lambda^i = u + (\mu^i)^2$, and $\mu^i,\ u$ satisfy the Gibbons-Tsarev system.

Deformations of one-phase reductions of dKP

$$\left(u_t - uu_x - \varepsilon^2 u_{xxx}\right)_x = u_{yy}$$

Deformed one-phase reductions (modulo the Miura group can assume $u = R$):

$$\begin{aligned} R_y =& \mu R_x \\ & + \varepsilon^2 \left(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3) R_x^2 \right)_x + O(\varepsilon^4) \\ R_t =& (\mu^2 + R) R_x \\ & + \varepsilon^2 \left((2\mu\mu' + 1) R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2) R_x^2 \right)_x + O(\varepsilon^4) \end{aligned}$$

Conjecture

For any soliton system in (2+1)D, all hydrodynamic reductions of its dispersionless limit can be deformed into reductions of the dispersive counterpart (linear non - degeneracy of the dispersionless limit is required).

Generalized KP equation

$$u_t - uu_x + \varepsilon(A_1 u_{xx} + A_2 u_x^2) + \varepsilon^2(B_1 u_{xxx} + B_2 u_x u_{xx} + B_3 u_x^3) = w_y$$

$$w_x = u_y$$

Require that all one-phase reductions can be deformed as

$$u = R, \qquad w = w(R) + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$

where

$$R_y = \mu R_x + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$

$$R_t = (\mu^2 + R)R_x + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$

$w' = \mu$. This gives $A_1 = A_2 = B_2 = B_3 = 0$, B_1=const, $\Longrightarrow$ KP

Classification result: scalar third order (2+1)D soliton equations with simplest nonlocalities

Based on *E.V. Ferapontov, A. Moro and V.S. Novikov, Integrable equations in* $2+1$ *dimensions: deformations of dispersionless limits, arXiv:0903.3586v1, to appear in J. Phys. A*

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\ldots) + \epsilon^2(\ldots), \quad w_x = u_y$$

here $\varphi,\ \psi,\ \eta$ are functions of u and w, and $(\ldots)$ denote terms which are polynomial in the derivatives of u and w with respect to x and y of orders 2 and 3, respectively. Here $w = D_x^{-1} D_y u$ is the nonlocality, no other non-local variables are allowed.

- Classify integrable dispersionless systems of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y$$

- Add dispersive corrections which inherit all hydrodynamic reductions (sufficient to consider 1-component reductions only)

Known examples

KP $$u_t = uu_x + w_y + \epsilon^2 u_{xxx}$$

mKP $$u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx}$$

Gardner $$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx}$$

VN $$u_t = (uw)_y + \epsilon^2 u_{yyy}$$

mVN $$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u} \right)_y$$

HarryDym $$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}$$

Classification of integrable dispersionless limits

Integrability conditions, based on *E.V. Ferapontov and K.R. Khusnutdinova, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen.* **37***, no. 8 (2004) 2949–2963.*

$$\varphi_{uu} = -\frac{\varphi_w^2 + \psi_u\varphi_w - 2\psi_w\varphi_u}{\eta}, \quad \varphi_{uw} = \frac{\eta_w\varphi_u}{\eta}, \quad \varphi_{ww} = \frac{\eta_w\varphi_w}{\eta}$$

$$\psi_{uu} = \frac{-\varphi_w\psi_w + \psi_u\psi_w - 2\varphi_w\eta_u + 2\eta_w\varphi_u}{\eta}, \quad \psi_{uw} = \frac{\eta_w\psi_u}{\eta}, \quad \psi_{ww} = \frac{\eta_w\psi_w}{\eta}$$

$$\eta_{uu} = -\frac{\eta_w\left(\varphi_w - \psi_u\right)}{\eta}, \quad \eta_{uw} = \frac{\eta_w\eta_u}{\eta}, \quad \eta_{ww} = \frac{\eta_w^2}{\eta}$$

In involution, straightforward to solve: $\eta = 1,\ \eta = u,\ \eta = e^w h(u)$

Conjecture

For any $\varphi,\ \psi,\ \eta$ one can reconstruct (non-uniquely) dispersive corrections which inherit all hydrodynamic reductions. Infinite series in ϵ are required in general.

New examples: 1

$$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2[B^3(u) - \beta B^2(u)u_x], \quad w_x = u_y,$$

$$B = \beta u D_x - D_y.$$

Lax pair

$$\psi_{xy} = \beta u \psi_{xx} + \frac{1}{3\epsilon^2}\psi,$$

$$\psi_t = \beta^3 \epsilon^2 u^3 \psi_{xxx} - \epsilon^2 \psi_{yyy} + 3\beta^2 \epsilon^2 u u_y \psi_{xx} + \beta w \psi_x$$

New examples: 2

$$u_t = \frac{4}{27}\gamma^2 u^3 u_x + (w+\gamma u^2)u_y + uw_y + \varepsilon^2\left[B^3(u) - \frac{1}{3}\gamma u_x B^2(u)\right], \quad w_x = u_y,$$

$B = \frac{1}{3}\gamma u D_x + D_y.$

Lax pair

$$\psi_{xy} = -\frac{\gamma}{3}u\psi_{xx} - \frac{1}{3\epsilon^2}u\psi,$$

$$\psi_t = \frac{\epsilon^2\gamma^3}{27}u^3\psi_{xxx} + \epsilon^2\psi_{yyy} - \frac{\epsilon^2\gamma^2}{3}uu_y\psi_{xx} + \frac{\gamma^2}{27}u^3\psi_x + w\psi_y - \frac{\gamma}{3}uu_y\psi$$

New examples: 3

$$u_t = \frac{\delta}{u^3} u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}, \quad w_x = u_y,$$

$\delta = 0$ gives the Harry Dym equation.

Lax pair $L_t = [A, L]$,

$$L = \frac{\epsilon^2}{u^2} D_x^2 + \frac{\epsilon}{\sqrt{3}} D_y + \frac{\delta^2}{4u^2},$$

$$A = \frac{4\epsilon^2}{u^3} D_x^3 + \left(-\frac{6\epsilon^2 u_x}{u^4} + \frac{2\sqrt{3}\epsilon w}{u^2}\right) D_x^2 + \frac{\delta}{u^3} D_x + \left(-\frac{3\delta u_x}{2u^4} + \frac{\sqrt{3}\delta w}{2\epsilon u^2}\right)$$

Comparison of 1+1 and 2+1 deformation schemes

1+1 case:

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + \varepsilon^2(\dots) + \dots$$

Dispersionless integrable systems form an infinite dimensional moduli space;

Terms at ε^2 contain extra functional freedom (central invariants).

2+1 case:

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + \varepsilon^2(\dots) + \dots$$

Dispersionless integrable systems form finite dimensional moduli spaces;

Terms at ε^2 contain no functional freedom.

There is a hope to obtain explicit formulae in 2+1D