Quasi-singular solitons and Alfvénic turbulence in the forced dissipative DNLS equation

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The Derivative NonLinear Schrödinger (DNLS) equation

\[ \frac{\partial b}{\partial t} + \alpha \frac{\partial}{\partial x} \left( |b|^2 b \right) + i\delta \frac{\partial^2 b}{\partial x^2} = \nu \frac{\partial^2 b}{\partial x^2} + f \]

- **Motivation 1:** one of the simplest models containing dispersion that can be used to study turbulence

\( b(x,t) \): complex field
\( \alpha, \delta, \nu \): real coefficients

Dispersive term
Dissipative term
Forcing
Motivation 2: DNLS as one of the simplest model to address the dynamics of magnetized interplanetary plasmas

Measured signals (power law spectra) suggest a turbulent environment. The dynamical origin of such power laws is still poorly understood.

Solar wind turbulent spectrum (Alexandrova et al., 2007)

Magnetic energy spectrum in the magnetosheath downstream of the bow shock (Alexandrova et al., JGR, 2006).
Where does DNLS come from in the space plasma context? Start from a large-scale fluid model for magnetized plasmas: Hall-MHD equations

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \]

\[ \rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\frac{\beta}{\gamma} \nabla \rho^\gamma + (\nabla \times \mathbf{b}) \times \mathbf{b} \]

\[ \partial_t \mathbf{b} - \nabla \times (\mathbf{u} \times \mathbf{b}) = -\frac{1}{R_i} \nabla \times \left( \frac{1}{\rho} (\nabla \times \mathbf{b}) \times \mathbf{b} \right) \]

\[ \nabla \cdot \mathbf{b} = 0 \]

\[ \beta = \frac{c_s^2}{c_A^2} \quad c_A \equiv \frac{B_0}{\sqrt{4\pi \rho_0}} = 1 \quad R_i = \frac{L}{d_i} \quad \text{where} \quad d_i \equiv \frac{c}{\omega_{pi}} = \frac{c_A}{\Omega_i} \]

\[ R_i = 1 \quad \text{when using the inertial length} \quad d_i \quad \text{as unit length} \]

MHD describes magnetized plasmas at large scales.

Fundamental scale introduced by the Hall term: dispersive scale, depending on plasma parameters.
Hall-MHD supports dispersive waves

Parallel-propagating Alfvén waves in Hall-MHD:
monochromatic, circularly polarized Alfvén waves are exact solutions

\[ b_y - \sigma ib_z = -\frac{\omega}{k} (u_y - \sigma i u_z) = B e^{i(kx - \omega t)} \]
\[ b_x = \rho = 1, \quad u_x = 0. \]

\( \sigma = \pm 1 \) for RH (LH) polarized waves

For forward propagation (k>0):
\[ \omega = \frac{\sigma k^2}{2 R_i} + k \sqrt{1 + \frac{k^2}{4 R_i^2}} \]

Alfvén waves are then dispersive
A long-wavelength reductive perturbative expansion leads to DNLS from Hall-MHD

\[ b \equiv b_y + \imath b_z + B_{0y} \]

- Transverse component of the ambient field \( B_0 \) making an angle with \( x \)

- Ambient field \( B_0 \)

- \( b \rightarrow \) direction of propagation of the waves (quasi-parallel or parallel propagation)

- Long-wavelength r. p. e. isolates the dynamics of Alfvén waves

- Weakly nonlinear, weakly dispersive regime: \( b, \nu \sim \epsilon^{1/2}, x \sim \epsilon, t \sim \epsilon^2 \)

- Slaved sonic waves: \( \rho \sim |b|^2 \)

\[ \frac{\partial b}{\partial t} + \alpha \frac{\partial}{\partial x} \left( |b|^2 b \right) + i \delta \frac{\partial^2 b}{\partial x^2} = 0 \]

DNLS can be rigorously derived from a fully kinetic plasma model (Vlasov) as well !!!
(Rogister, 1971)
Soliton solutions of (unforced, undissipated) DNLS

The unforced, undissipated DNLS equations is integrable by inverse scattering and possesses soliton solutions (Kaup and Newell, 1978)

Parallel-propagating soliton: \( b = B \exp(i\theta) \)

\[
B = B(x - vt)
\]

\[
B^2 = \frac{4(\nu_0 + \kappa_0^2)}{(\nu_0 + 2\kappa_0^2)^{1/2} \cosh[(x - vt - x_0)/\delta] + \kappa_0}
\]

\[
\kappa = \kappa_0 + (3/4)B^2, \quad \nu = \nu_0 + (3/4)vB^2
\]

\[
\nu = -\theta_t \quad \kappa = \theta_x
\]
Obliquely-propagating soliton:
characterized by an eigenvalue $\lambda$ and an amplitude $b_0$ at infinity

$\lambda$ real: bright or dark solitons

$$b(x, t) = b_0 \pm 2 \frac{(1 - \frac{\eta}{\lambda} i) \exp[-2\Gamma(x - ct)]}{\left\{1 + \frac{b_0}{2\eta^2} \left(1 - \frac{\eta}{\lambda} i\right) \exp[-2\Gamma(x - ct)]\right\}^2}$$

$$c = 2\lambda^2 + b_0^2 \quad \eta = \sqrt{\lambda^2 - b_0^2}$$

$$0 < \lambda < b_0 \quad (=1) \quad \Gamma = \eta \lambda$$

$\lambda$ complex: “breathers”
(solitons whose amplitude oscillates during the propagation)
Facts and questions

• Forced – dissipated DNLS tends to develop solitons which undergo a “quasi-collapse” dynamics.

• Quasi-collapse soliton dynamics can be reproduced by adding a weak dissipation to an existing soliton. What are the properties of this phenomenon?

• Do forced-dissipated DNLS show (dispersive) turbulence? What does this turbulence look like? What is then the role of solitons?

Much of this is still work in progress!!!
DNLS + random forcing and $\partial_{xx}$ dissipation:
« turbulent » behaviour and strongly peaked solitons

$$\frac{\partial b}{\partial t} + a \frac{\partial}{\partial x} (|b|^2 b) + i\delta \frac{\partial^2 b}{\partial x^2} = \nu \frac{\partial^2 b}{\partial x^2} + f$$

Large-scale random forcing, white in time

1) Soliton accelerates, becomes thinner and reverse propagation direction
2) Soliton crashes to a wavepacket
Solitons spontaneously form in randomly forced DNLS and are dissipated in a dramatic way. We investigated the dynamics of DNLS solitons subject to dissipation only.

We start with an oblique bright soliton with $\lambda$ real

$$b(x, t) = b_0 \pm 2 \frac{\left(1 - \frac{n}{\lambda} i\right) \exp \left[-2\Gamma(x - ct)\right]}{\left\{1 \mp \frac{b_0}{2\eta^2} \left(1 - \frac{n}{\lambda} i\right) \exp \left[-2\Gamma(x - ct)\right]\right\}^2}$$

$$c = 2\lambda^2 + b_0^2 \quad \eta = \sqrt{\lambda^2 - b_0^2} \quad \cdot < \lambda < b. \quad (=1) \quad \Gamma = \eta \lambda$$

We let the soliton evolve subject to a weak dissipation only: no forcing

Either standard viscosity: $\nu \partial_{xx}^2 b$

or nonlinear dissipative term: $\nu \partial_x H |b|^2 b$ (Landau damping)
Soliton evolution in time

\[ \lambda = 0.97 \text{ real} \]

\[ \nu = 10^{-5} \]

Soliton undergoes a "quasi-collapse" leaving behind dark solitons, then reverses its direction of propagation and evolves to a wavepacket up to 32000 mesh points.
How does this “quasi-collapse” depend on viscosity?

Time evolution of the soliton amplitude $|b|$ for different viscosities

$|b|(t)$

$\nu=10^{-3}$  $\nu=10^{-4}$  $\nu=10^{-5}$

$\lambda = \cdot .9 \forall$ real

Not inconsistent with a singularity in the limits $t \to \infty, \nu \to 0$
The problem of weakly dissipative DNLS is in fact a long-standing one.

DNLS solitons are characterized by an eigenvalue $\lambda$ (giving for example the soliton speed).

For weakly perturbed DNLS, $\lambda$ will be (hopefully) a slowly varying time variable: $\lambda = \lambda(t)$ (see e.g. Laskhin 2007 for a review).

Changing viscosity should result in a trivial time-rescaling: $t \rightarrow t/\nu$
Maximal amplitude does not depend on viscosity: just $|b|(t^*\nu)$

In any case, perturbative theory cannot explain oscillations.

Here it does not seem the case (or the viscosity must be taken too small for the asymptotics to be numerically accessible).
Well, in fact the behaviour depends on which initial soliton we consider (i.e. the value of \( \lambda \)).

We took \( \lambda \) complex with different imaginary parts and plot \(|b|(t^* \nu)\) with 2 (or 3) different viscosities.

Here it works like in perturbation theory, apart from oscillations.

Going towards the real axis perturbation theory is less and less good.
When “collapsing”, the soliton leaves behind a train of “depressions” (dark solitons ?)

Hamilton et al. (JGR 114, A03104, 2009) at large viscosity:
the soliton bifurcates.

\( \lambda \) real -> bifurcation to \( \lambda \) complex with emission of a dark soliton.
This bifurcation is not captured by perturbation theory.

This was a study at large fixed viscosity, discarding any limit effect as \( \nu \to 0 \).
We are working to verify and extend this interpretation at small viscosity.
What happens when \( \lambda \) starts as complex?
Evolution of $\lambda$ as a function of time

Starting with real $\lambda$: bifurcation to complex

- Experimental behaviour as $\nu \to 0$?
- Mathematical theory of all that is still lacking

Starting with complex $\lambda$
The soliton ultimate destiny

After quasi-collapse, the soliton evolves toward a wavepacket-like structure + a hole

Larger view
This formation and disruption of solitons is very similar to what happens in randomly forced DNLS.

Forced case

Unforced case

Creation and disruption of solitons appears as one of the building-blocks of forced DNLS phenomenology.
Forced – dissipated DNLS: a more complete phenomenology.

Turbulence?

\[
\frac{\partial b}{\partial t} + \alpha \frac{\partial}{\partial x} \left( |b|^2 b \right) + i\delta \frac{\partial^2 b}{\partial x^2} = \nu \frac{\partial^2 b}{\partial x^2} + f
\]

• Due to integrability, resonant couplings are absent: no weak turbulence (in the unforced, undissipated case)

• Still a possibility for strong dispersive turbulence

First of all: what happens if we turn off dispersion? (no Hall effect \(\delta=0\): Cohen-Kulsrud equation (Cohen, Kulsrud, ...)

\[
\frac{\partial b}{\partial t} + \alpha \frac{\partial}{\partial x} \left( |b|^2 b \right) = \nu \frac{\partial^2 b}{\partial x^2} + f
\]

Kind of “cubic Burgers equation” for a complex field \(b\)

Burgers equation for a real field \[
\frac{\partial u}{\partial t} + \alpha \frac{\partial (u^2)}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f
\]
Cohen-Kulsrud (no dispersion) with different viscosities $\nu$ and same forcing

$\nu=2.5e-3$ (red), $1.e-3$ (yellow), $4.e-4$ (green), $2.e-4$ (light blue), $1.e-4$ (black)

Power law energy spectra
Fronts and sharp discontinuities
Growth of mean $|b|^2$
Larger structures for smaller $\nu$ (viscosity influences large-scale properties)
Now add dispersion: true DNLS

\[ \frac{\partial b}{\partial t} + \alpha \frac{\partial}{\partial x} (|b|^2 b) + i\delta \frac{\partial^2 b}{\partial x^2} = \nu \frac{\partial^2 b}{\partial x^2} + f \]

Viscosity \( \nu = 10^{-7} \)

Dispersion \( \delta = 1 \cdot 10^{-3} \)

\( \nu / \delta = 10^{-4} \ll 1 \)

Shock-like structures develop dispersive oscillations

Solitons appear
Increase dispersion

Viscosity \( \nu = 10^{-7} \)

Dispersion \( \delta = 5 \cdot 10^{-3} \)

\( \nu / \delta = 2 \cdot 10^{-3} \ll 1 \)

1) Components: fronts + dispersive oscillations
2) \( |b|^2 \): many soliton-like structures
3) A well-defined power law energy spectrum
4) Transient solitons
5) Generation of a mean $|b|^2$. Large-scale fronts (coherent structures) are characterized by an almost-constant $|b|^2$. (nl term $b|b|^2$ is depleted)

Energy can grow without limit, due to mean $|b|^2$, so it is not a good measure of turbulence level and stationarity. We can use the variance of $|b|^2$ to recognize the stationary state.

Variance($|b|^2$)
for $v=2\cdot10^{-4}$, $1\cdot10^{-5}$, $1\cdot10^{-7}$
Stronger viscosity: looks like dispersive shocks instead

Dispersive structures
Power-law is not associated with a constant energy flux in spectral space (unlike standard turbulence).

\[
\frac{\nu}{\delta} = 1
\]

\[
\frac{\nu}{\delta} = 10^{-2}
\]

\[
\frac{\nu}{\delta} = 10^{-4}
\]

Instantaneous flux \( F(k) \) for \( \delta = 5 \cdot 10^{-3} \) and three different viscosities.

At large enough \( \nu \), the shape of the energy flux suggests the existence of an inertial range.

Decreasing \( \nu \), the flux becomes wildly intermittent (consistent of large fluctuating spikes).

-> no orderer energy cascade exists in the dispersive regime, unlike standard turbulence.
Large transient solitons are associated with intense dissipation bursts. Mean dissipation is still consistent with a finite limit as $\nu \to 0$, like in standard turbulence.

To resume: energy flux is wildly influenced by dispersive structures, being dominated by strongly intermittent events. Conversely, integral dissipation behaves like in standard turbulence but big transient solitons are conspicuous.

A word of caution about numerics: with an explicit scheme (RK3) you must take very tiny timesteps (10-20 times smaller than the stability limit) to accurately capture dissipation with very low viscosity. You definitely need spectral methods, moreover!
To resume

Forced – dissipated DNLS equation: a model for space plasmas supporting dispersive waves

- Spontaneous generation of solitons under random forcing.
- Solitons evolve toward smaller scales and strong amplitudes under the effect of a small dissipation (quasi-collapse), before dramatically disappearing.
- The smaller the dissipation, the stronger the amplification (depending on the initial soliton).
- A bifurcation mechanism can be responsible for the solitons quasi-collapse. In the limit of small viscosity, further investigations are in progress.
- In the forced regime, a 'many-soliton-like' regime is generated if the viscosity is sufficiently low. There are no dispersive shocks anymore.
- In this regime, there is no definite constant energy flux, differently from standard turbulence.
- Integrated dissipation is consistent with a finite limit as \( \nu \to 0 \), like in standard turbulence.
- Big transient solitons are a major feature, responsible for bursts of dissipation.