On the relationship between nonlinear equations integrable by the method of characteristics and equations associated with commuting vector fields

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2. Relationship between nonlinear equations integrable by the method of characteristics and equations associated with commuting vector fields:
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1. Introduction
2. Relationship between nonlinear equations integrable by the method of characteristics and equations associated with commuting vector fields
(a) General scheme of transformations

(1+1) dimensional matrix PDE integrable by the method of characteristics (Ch-integrable PDE):

\[ W_{tn} + W_{zn} W = 0, \quad n = 1, 2. \]

Simple example of the scalar nonlinear PDE associated with commuting vector fields.

\[ u_{z_1 t_2} - u_{z_2 t_1} + u_{z_2} u_{z_1 x} - u_{z_1} u_{z_2 x} = 0, \]

Lax pair:

\[ \psi_{t_i}(\lambda; \vec{x}) + \lambda \psi_{z_i}(\lambda; \vec{x}) + u_{z_i}(\vec{x}) \psi_x(\lambda; \vec{x}) = 0, \quad i = 1, 2 \]

\[ \vec{x} = (z_1, z_2, t_1, t_2, x) \]
The chain of transformations from the (1+1)-dimensional PDE integrable by the method of characteristics to the nonlinear PDE associated with commuting vector fields; $N^{S_2} = 2n_0M$
2(b). Derivation of $GL(N)$ SDYM

The linear system

\[ \chi \Lambda = W \chi, \]
\[ \chi_{t_n} + \chi_{z_n} \Lambda = 0, \quad n = 1, 2, \ldots. \]

\(\chi\) and \(W\) are \(2Mn_0N_0 \times 2Mn_0N_0\) matrix functions and \(\Lambda\) is a diagonal \(2Mn_0(N_0 + 1) \times 2Mn_0N_0\) matrix function. Parameters \(M, n_0\) and \(N_0\) are arbitrary integers.

Compatibility condition:

\[ \Lambda_{t_n} + \Lambda_{z_n} \Lambda = 0, \]
\[ W_{t_n} + W_{z_n} W = 0. \]
Frobenius structure of $w$:

$$W = \begin{pmatrix}
W^{(1)} & W^{(2)} & \cdots & W^{(N_0-1)} & W^{(N_0)} \\
I_{2Mn_0} & 0_{2Mn_0} & \cdots & 0_{2Mn_0} & 0_{2Mn_0} \\
0_{2Mn_0} & I_{2Mn_0} & \cdots & 0_{2Mn_0} & 0_{2Mn_0} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0_{2Mn_0} & 0_{2Mn_0} & \cdots & I_{2Mn_0} & 0_{2Mn_0}
\end{pmatrix},$$

Block-diagonal form of $\Lambda$:

$$\Lambda = \text{diag}(\Lambda^{(1)}, \ldots, \Lambda^{(2N_0)}),$$

and

$$\chi = \begin{pmatrix}
\chi^{(1)} & \cdots & \chi^{(N_0)} \\
\chi^{(1)}(\Lambda^{(1)})^{-1} & \cdots & \chi^{(N_0)}(\Lambda^{(N_0)})^{-1} \\
\cdots & \cdots & \cdots \\
\chi^{(1)}(\Lambda^{(1)})^{-N_0+1} & \cdots & \chi^{(N_0)}(\Lambda^{(N_0)})^{-N_0+1}
\end{pmatrix},$$

where $W^{(i)}$ and $\chi^{(i)}$ are $2Mn_0 \times 2Mn_0$ matrix functions, $\Lambda^{(j)}$ are $2Mn_0 \times 2Mn_0$ diagonal matrices.
Discrete chain:

\[ W_{tn} + W_{zn} W = 0 \quad \Rightarrow \quad W^{(i)}_{tn} + W^{(1)}_{zn} W^{(i)} + W^{(i+1)} = 0, \]

\[ i = 1, \ldots, N_0, \quad W^{(N_0+1)} = 0, \quad n = 1, 2. \]

Let \( i = 1 \) and eliminate \( W^{(2)} \) to obtain self-dual Yang-Mills equation:

\[ W^{(1)}_{z_1 t_2} - W^{(1)}_{z_2 t_1} + W^{(1)}_{z_2} W^{(1)}_{z_1} - W^{(1)}_{z_1} W^{(1)}_{z_2} = 0, \]
2(c) $GL(N)$ SDYM and differential reduction

Let
\[
\chi^{(j)} = \begin{pmatrix}
\Psi^{(2j-1)} & \Psi^{(2j)} \\
\Psi_x^{(2j-1)} & \Psi_x^{(2j)}
\end{pmatrix}, \quad \Lambda^{(j)} = \text{diag}(\tilde{\Lambda}^{(2j-1)}, \tilde{\Lambda}^{(2j)}), \quad j = 1, \ldots, N_0,
\]

where $\Psi^{(m)}$ are $Mn_0 \times Mn_0$ matrix functions:
\[
\Psi^{(m)}_{xx} = a\Psi^{(m)}\tilde{\Lambda}^{(m)} + \nu\Psi^{(m)}_{x} + \mu\Psi^{(m)}, \quad m = 1, \ldots, 2N_0,
\]

Then the linear eq. for $\Psi$ yields
\[
\Psi^{(j)}\Lambda^{(j)} = \sum_{i=0}^{N_0} W^{(i)}\Psi^{(j)}(\Lambda^{(j)})^{-i}, \quad j = 0, 1, \ldots, N_0,
\]

where $a$, $\nu$ and $\mu$ are $n_0M \times n_0M$ diagonal constant matrix parameters and
\[
W^{(i)} = \begin{pmatrix}
w^{(i)} & v^{(i)} \\
w_x^{(i)} + v^{(i)}\mu + v^{(i+1)}a + v^{(0)}aw^{(i)} & v_x^{(i)} + v^{(i)}\nu + w^{(i)} + v^{(0)}av^{(i)}
\end{pmatrix}.
\]
The first block-row reads:

\[ \Psi \Lambda = \sum_{i=1}^{N_0} \left( v^{(i)} \Psi_x + w^{(i)} \Psi \right) \Lambda^{-i+1}. \]

\( \Psi \) is a solution of the system

\[ \Psi_t + \Psi_z \Lambda = 0, \]
\[ \mathcal{E}^{(0)} := \Psi_{xx} = a \Psi \Lambda + \nu \Psi_x + \mu \Psi, \]

where

\[ \Psi = (\Psi^{(1)}, \ldots, \Psi^{(2N_0)}), \]
\[ \Lambda = \text{diag}(\tilde{\Lambda}^{(1)}, \ldots, \tilde{\Lambda}^{(2N_0)}), \]

\( v^{(i)}, w^{(i)}, \Psi^{(i)} \) and \( \tilde{\Lambda}^{(j)} \) are \( Mn_0 \times Mn_0 \) matrix functions, \( a, \nu \) and \( \mu \) are \( Mn_0 \times Mn_0 \) diagonal constant matrices.
Compatibility conditions:

$$\Lambda_t + \Lambda_z \Lambda = 0, \quad \Lambda_x = 0$$

Evolution part of the nonlinear system

$$E_n^{(1i)} := v_{tn}^{(i)} + v_{zn}^{(1)}(w^{(i)} + v_x^{(i)} + v^{(i)} \nu + v^{(1)} a v^{(i)}) + w_{zn}^{(1)} v^{(i)} + v_{zn}^{(i+1)} = 0,$$

$$v^{(N_0+1)} = 0, \quad i = 1, \ldots, N_0,$$

$$E_n^{(0i)} := w_{tn}^{(i)} + v_{zn}^{(1)}(w_x^{(i)} + v^{(i)} \mu + v^{(1)} a w^{(i)} + v^{(i+1)} a) + w_{zn}^{(1)} w^{(i)} + w_{zn}^{(i+1)} = 0,$$

$$w^{(N_0+1)} = 0, \quad i = 1, \ldots, N_0,$$

Non-evolution part of the nonlinear system:

$$\tilde{E}^{(1i)} := v_{xx}^{(i)} + [w^{(i)}, \nu] + 2w_x^{(i)} + 2v_x^{(i)} \nu - \nu v_x^{(i)} + [v^{(i)}, \mu] + [v^{(i)}, \nu] \nu + [v^{(1)}, \nu] a v^{(i)} + 2v_x^{(1)} a v^{(i)} + [w^{(1)}, a] v^{(i)} + [v^{(1)}, a] (w^{(i)} + v_x^{(i)} + v^{(i)} \nu + v^{(1)} a v^{(i)}) + [v^{(i+1)}, a] = 0,$$

$$\tilde{E}^{(0i)} := w_{xx}^{(i)} + [w^{(i)}, \mu] + [w^{(1)}, a] w^{(i)} + 2v_x^{(i)} \mu + [v^{(i)}, \nu] \mu + [v^{(1)}, \nu] a w^{(i)} + 2v_x^{(1)} a w^{(i)} - \nu w_x^{(i)} + [v^{(1)}, a] (w_x^{(i)} + v^{(i)} \mu + v^{(1)} a w^{(i)} + v^{(i+1)} a) + [w^{(i+1)}, a] + 2v_x^{(i+1)} a + [v^{(i+1)}, \nu] a = 0.$$
The complete system of PDEs for $v^{(i)}$ and $w^{(i)}$, $i = 1, 2$ corresponds to $i = 0$:

$$E_{n}^{(11)} := v_{tn}^{(1)} + v_{zn}^{(1)}(w^{(1)} + v_{x}^{(1)} + v^{(1)}\nu + v^{(1)}aw^{(1)}) + w_{zn}^{(1)}v^{(1)} + v_{zn}^{(2)} = 0,$$

$$E_{n}^{(01)} := w_{tn}^{(1)} + v_{zn}^{(1)}(w_{x}^{(1)} + v^{(1)}\mu + v^{(1)}aw^{(1)} + v^{(2)}a) + w_{zn}^{(1)}w^{(1)} + w_{zn}^{(2)} = 0,$$

$$\tilde{E}_{n}^{(11)} := v_{xx}^{(1)} + [w^{(1)}, \nu] + 2w_{x}^{(1)} + 2v_{x}^{(1)}\nu - \nu v_{x}^{(1)} + [v^{(1)}, \mu] + [v^{(1)}, \nu](\nu + av^{(1)}) + 2v_{x}^{(1)}aw^{(1)} + [w^{(1)}, a]v^{(1)} + [v^{(1)}, a](w^{(1)} + v_{x}^{(1)} + v^{(1)}\nu + v^{(1)}aw^{(1)}) + [v^{(2)}, a] = 0,$$

$$\tilde{E}_{n}^{(01)} := w_{xx}^{(1)} + [w^{(1)}, \mu] + [w^{(1)}, a]w^{(1)} + 2v_{x}^{(1)}\mu + [v^{(1)}, \nu]\mu + [v^{(1)}, \nu]aw^{(1)} + 2v_{x}^{(1)}aw^{(1)} - \nu w_{x}^{(1)} + [v^{(1)}, a](w_{x}^{(1)} + v^{(1)}\mu + v^{(1)}aw^{(1)} + v^{(2)}a) + [w^{(2)}, a] + 2v_{x}^{(2)}a + [v^{(2)}, \nu]a = 0$$
2(d). Frobenius reduction and associated higher dimensional systems of nonlinear PDEs

\[ v^{(i)} = \left\{ v^{(i;kl)}, \ k, l = 1, \ldots, n_0 \right\}, \quad w^{(i)} = \left\{ w^{(i;kl)}, \ k, l = 1, \ldots, n_0 \right\}, \]

\[ v^{(i;kl)} = \delta_{k1} v^{(i;l)} + \delta_{k(l+n_1(i))} I_M, \]

\[ w^{(i;kl)} = \delta_{k1} w^{(i;l)} + \delta_{k(l+n_2(i))} I_M, \]

where \( n_0 \) is an integer parameter, \( v^{(i;l)} \) and \( w^{(i;l)} \) are \( M \times M \) matrix fields, \( \nu, \mu \) and \( a \) have the following diagonal forms:

\[ \nu = \text{diag}(\tilde{\nu}, \ldots, \tilde{\nu}) \]
\[ \mu = \text{diag}(\tilde{\mu}, \ldots, \tilde{\mu}) \]
\[ a = \text{diag}(\tilde{a}, \ldots, \tilde{a}) \]

where \( \tilde{\nu}, \tilde{\mu} \) and \( \tilde{a} \) are \( M \times M \) diagonal matrices.

Block structure of the nonlinear PDEs:

\[ E_n^{(mi)} = \left\{ E_n^{(mi;l)} \delta_{k1}, \ k, l = 1, \ldots, n_0 \right\}, \]

\[ \tilde{E}_n^{(mi)} = \left\{ \tilde{E}_n^{(mi;l)} \delta_{k1}, \ k, l = 1, \ldots, n_0 \right\}, \ m = 0, 1, \]
Explicit form of the nonlinear PDEs, \( n_1(i) = n_2(i) = 1 \)

\[
E_n^{(11;1)} := v_{i_n}^{(1;1)} + v_{z_n}^{(1;1)}(w^{(1;1)} + v_x^{(1;1)} + v^{(1;1)} + (v_{z_n}^{(1;1)} v^{(1;1)} + v_{z_n}^{(1;1+n_1(0))} \tilde{a}) + w_{z_n}^{(1;1)} v^{(1;1)} + (Q_1^{(1;1)})_{z_n} = 0,
\]

\[
E_n^{(01;1)} := w_{t_n}^{(1;1)} + v_{z_n}^{(1;1)}(w_x^{(1;1)} + v^{(1;1)} \mu + v^{(2;1)} \tilde{a}) + (v_{z_n}^{(1;1)} v^{(1;1)} + v_{z_n}^{(1;1+n_1(0))} \tilde{a}) + w_{z_n}^{(1;1)} w^{(1;1)} + (Q_2^{(1;1)})_{z_n} = 0,
\]

\[
\tilde{E}^{(11;l)} := \tilde{F}^{(11;l)}(v^{(1;1)}, w^{(1;1)}, v^{(1;2)}) + [Q_1^{(1;1)}, \tilde{a}],
\]

\[
\tilde{E}^{(01;l)} := \tilde{F}^{(11;l)}(v^{(1;1)}, w^{(1;1)}, v^{(2;1)}, v^{(1;2)}) + [Q_2^{(1;1)}, \tilde{a}]
\]

where

\[
Q_1^{(1;1)} = v^{(1;3)} \tilde{a} + v^{(1;2)} + v^{(2;1)} \tilde{\nu} + w^{(1;2)} + v^{(2;1)},
\]

\[
Q_2^{(1;l)} = v^{(1;3)} \tilde{a} + v^{(1;2)} \tilde{\mu} + v^{(1;2)} a + w^{(1;2)} + w^{(2;1)}.
\]
The complete system of PDEs for the matrix fields $v^{(1;1)}$, $w^{(2;1)}$, $v^{(2;1)}$, $v^{(1;2)}$:

\[
(E^{(11;1)}_1)_{z_2} - (E^{(11;1)}_2)_{z_1} = 0,
\]

\[
(E^{(01;1)}_1)_{z_2} - (E^{(01;1)}_2)_{z_1} = 0,
\]

\[
[E^{(11;1)}_1, \tilde{a}] - (\tilde{E}^{(11;1)})_{z_1} = 0,
\]

\[
[E^{(01;1)}_1, \tilde{a}] - (\tilde{E}^{(01;1)})_{z_1} = 0.
\]

The scalar case: $u \equiv v^{(1;1)}$

\[
(E^{(11;1)}_1)_{z_2} - (E^{(11;1)}_2)_{z_1} = 0 \iff u_{z_1} t_2 - u_{z_2} t_1 + u_{z_2} u_{z_1} x - u_{z_1} u_{z_2} x = 0.
\]
2(e). Solutions
Starting equation, $M = 1$:

$$\chi \Lambda = W \chi$$

After all reductions, this matrix equation may be splitted into two subsystems of scalar equations:

First subsystem:

$$\Psi^{(l;1m)} \tilde{\Lambda}^{(l;m)} = \sum_{i=1}^{N_0} \sum_{j=1}^{n_0} \left[ v^{(i;j)} \Psi^{(l;jm)} + w^{(i;j)} \Psi^{(l;jm)} \right] (\tilde{\Lambda}^{(l;m)})^{-i+1},$$

$$l = 1, \ldots, 2N_0, \ m = 1, \ldots, n_0.$$ 

This is the system of $2N_0n_0$ linear algebraic equations for the same number of matrix fields $v^{(i;l)}$ and $w^{(i;l)}$, $i = 1, \ldots, N_0$, $j = 1, \ldots, n_0$.

Second subsystem: $n > 1$,

$$\Psi^{(l;nm)} \tilde{\Lambda}^{(l;m)} = \sum_{i=0}^{N_0} \left[ \Psi^{(l;(n-n_1(i))m)} + \Psi^{(l;(n-n_2(i))m)} \right] (\tilde{\Lambda}^{(l;m)})^{-i+1},$$

$$\Psi^{(l;ij)} = 0, \text{ if } i \leq 0,$$

$$l = 1, \ldots, 2N_0, \ n, m = 1, \ldots, n_0.$$ 

This equation expresses recursively the functions $\Psi^{(l;nm)}$, $n > 1$, in terms of the functions $\Psi^{(l;1m)}$ and their $x$-derivatives.
Functions $\Psi^{(l;nm)}$ and $\tilde{\Lambda}^{(l;m)}$ are defined as follows:

$$
\Psi^{(l;1m)}(\vec{x}) = \sum_{i=1}^{2} \psi^{(lm;i)}(z_1 - \tilde{\Lambda}^{(l;m)} t_1, z_2 - \tilde{\Lambda}^{(l;m)} t_2) e^{k^{(lm;i)} x}
$$

$$
\tilde{\Lambda}^{(l;m)} = E^{(lm)}(z_1 - \tilde{\Lambda}^{(l;m)} t_1, z_2 - \tilde{\Lambda}^{(l;m)} t_2),
$$

$$
k^{(lm;1)} = \frac{1}{2} \left( \tilde{\nu} + \sqrt{\tilde{\nu}^2 + 4a\tilde{\Lambda}^{(l;m)} + 4\tilde{\mu}} \right), \quad k^{(lm;1)} = \frac{1}{2} \left( \tilde{\nu} - \sqrt{\tilde{\nu}^2 + 4a\tilde{\Lambda}^{(l;m)} + 4\tilde{\mu}} \right),
$$

$m = 1, \ldots, n_0, \quad l = 1, \ldots, 2N_0.$
Simplest example of the solution: \( N_0 = n_0 = 1 \)

\[
\Psi^{(l;11)}(\vec{x}) = \sum_{i=1}^{2} \psi^{(l;1)}(z_1 - \tilde{\Lambda}^{(l;1)}t_1, z_2 - \tilde{\Lambda}^{(l;1)}t_2)e^{k^{(l;1)}x}, \; l = 1, 2,
\]

\[
\tilde{\Lambda}^{(l;1)} = E^{(l1)}(z_1 - \tilde{\Lambda}^{(l;1)}t_1, z_2 - \tilde{\Lambda}^{(l;1)}t_2), \; l = 1, 2.
\]

Then

\[
u \equiv v^{(1;1)} = \frac{\Delta_1}{\Delta},
\]

\[
\Delta = \sum_{i,j=1}^{2} \left[ (-1)^{i+1}K_1 - (-1)^{j+1}K_2 \right]e^{\left( \tilde{\nu} + (-1)^{i+1}K_1 + (-1)^{j+1}K_2 \right)x}\psi^{(11;1)}(y_1^1, y_1^2)\psi^{(21;1)}(y_2^1, y_2^2),
\]

\[
K_l = \frac{1}{2} \sqrt{\tilde{\nu}^2 + 4\tilde{a}\tilde{\Lambda}^{(l;1)} + 4\tilde{\mu}}, \; y_1^1 = z_1 - \tilde{\Lambda}^{(l;1)}t_1, \; y_2^1 = z_2 - \tilde{\Lambda}^{(l;1)}t_2,
\]

\[
\Delta_1 = (\tilde{\Lambda}^{(1;1)} - \tilde{\Lambda}^{(2;1)}) \sum_{i,j=1}^{2} e^{\left( \tilde{\nu} + (-1)^{i+1}K_1 + (-1)^{j+1}K_2 \right)x}\psi^{(11;1)}(y_1^1, y_1^2)\psi^{(21;1)}(y_2^1, y_2^2)
\]

Solution \( u \) has no singularities if \( \Delta \neq 0: K_1 > K_2 > 0, \; \psi^{(11;2)} < 0, \; \psi^{(11;1)}, \psi^{(21;1)}, \psi^{(21;2)} > 0. \)

Since \( \tilde{\Lambda}^{(l;1)} \) is implicitly given by the eq.(1), constructed function \( u \) describes the break of the wave profile unless \( \tilde{\Lambda}^{(l;1)} = const. \)
For instance:

\[ K_2 = 0 \implies \tilde{\Lambda}^{(2;1)} = -\frac{\tilde{\nu}^2 + 4\tilde{\mu}}{4\tilde{a}} = const, \]

\[ \psi^{(11;1)}(y_1^1, y_2^1) = \xi_1(y_1^1, y_2^1) > 0, \quad \psi^{(11;2)}(y_1^2, y_2^2) = -\xi_2(y_1^2, y_2^2) < 0. \]

One has

\[ u = \frac{(\tilde{\Lambda}^{(1;1)} - \tilde{\Lambda}^{(2;1)}) \left( e^{K_1 x} \xi_1(y_1^1, y_2^1) - e^{-K_1 x} \xi_2(y_1^2, y_2^2) \right)}{K_1 \left( e^{K_1 x} \xi_1(y_1^1, y_2^1) + e^{-K_1 x} \xi_2(y_1^2, y_2^2) \right)}, \]

\[ y_1^l = z_1 - \tilde{\Lambda}^{(l;1)} t_1, \quad y_2^l = z_2 - \tilde{\Lambda}^{(l;1)} t_2, \]