On the relationship between nonlinear equations integrable by the method of characteristics and equations associated with commuting vector fields

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2. Relationship between nonlinear equations integrable by the method of characteristics and equations associated with commuting vector fields:
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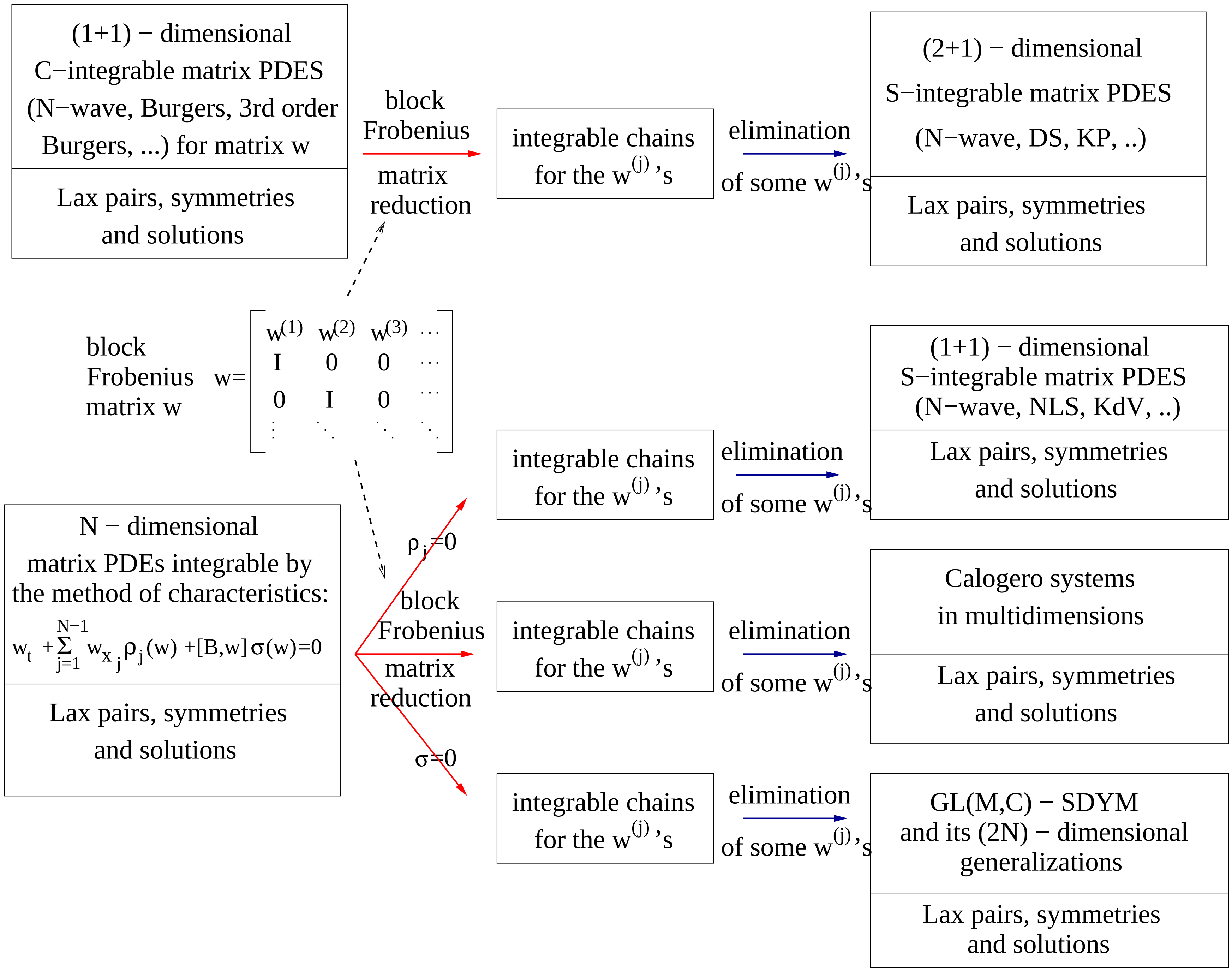
1. Introduction

A.I.Zenchuk and P.M.Santini, *J.Phys.A:Math.Theor.*, **41** (2008) 185209



2. Relationship between nonlinear equations integrable by the method of characteristics and equations associated with commuting vector fields

A. I. Zenchuk, J.Math.Phys., **50** 063505 (2009)

(a) General scheme of transformations

(1+1) dimensional matrix PDE integrable by the method of characteristics (Ch-integrable PDE):

$$W_{t_n} + W_{z_n} W = 0, \quad n = 1, 2.$$

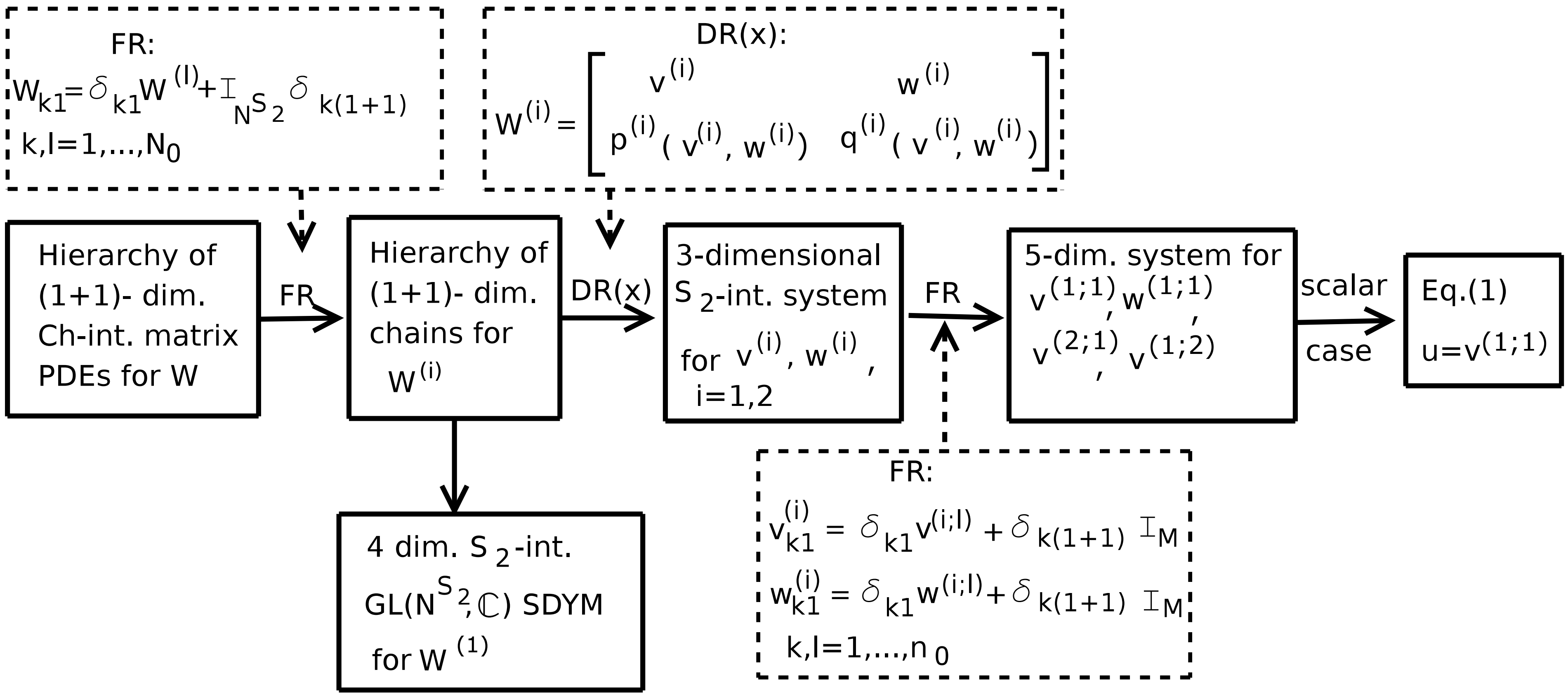
Simple example of the scalar nonlinear PDE assotiated with commuting vector fields.

$$u_{z_1 t_2} - u_{z_2 t_1} + u_{z_2} u_{z_1 x} - u_{z_1} u_{z_2 x} = 0,$$

Lax pair:

$$\psi_{t_i}(\lambda; \vec{x}) + \lambda \psi_{z_i}(\lambda; \vec{x}) + u_{z_i}(\vec{x}) \psi_x(\lambda; \vec{x}) = 0, \quad i = 1, 2$$

$\vec{x} = (z_1, z_2, t_1, t_2, x)$



The chain of transformations from the (1+1)-dimensional PDE integrable by the method of characteristics to the nonlinear PDE assotiated with commuting vector fields; $N^{S_2} = 2n_0M$

2(b). Derivation of $GL(N)$ SDYM

The linear system

$$\chi\Lambda = W\chi,$$
$$\chi_{t_n} + \chi_{z_n}\Lambda = 0, \quad n = 1, 2, \ldots .$$

χ and W are $2Mn_0N_0 \times 2Mn_0N_0$ matrix functions and Λ is a diagonal $2Mn_0(N_0+1) \times 2Mn_0N_0$ matrix function. Parameters M, n_0 and N_0 are arbitrary integers.

Compatibility condition:

$$\Lambda_{t_n} + \Lambda_{z_n}\Lambda = 0,$$
$$W_{t_n} + W_{z_n}W = 0.$$

Frobenius structure of w:

$$W = \begin{pmatrix} W^{(1)} & W^{(2)} & \cdots & W^{(N_0-1)} & W^{(N_0)} \\ I_{2Mn_0} & 0_{2Mn_0} & \cdots & 0_{2Mn_0} & 0_{2Mn_0} \\ 0_{2Mn_0} & I_{2Mn_0} & \cdots & 0_{2Mn_0} & 0_{2Mn_0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{2Mn_0} & 0_{2Mn_0} & \cdots & I_{2Mn_0} & 0_{2Mn_0} \end{pmatrix},$$

Block-diagonal form of Λ:

$$\Lambda = \mathrm{diag}(\Lambda^{(1)}, \ldots, \Lambda^{(2N_0)}),$$

and

$$\chi = \begin{pmatrix} \chi^{(1)} & \cdots & \chi^{(N_0)} \\ \chi^{(1)}(\Lambda^{(1)})^{-1} & \cdots & \chi^{(N_0)}(\Lambda^{(N_0)})^{-1} \\ \cdots & \cdots & \cdots \\ \chi^{(1)}(\Lambda^{(1)})^{-N_0+1} & \cdots & \chi^{(N_0)}(\Lambda^{(N_0)})^{-N_0+1} \end{pmatrix},$$

where $W^{(i)}$ and $\chi^{(i)}$ are $2Mn_0 \times 2Mn_0$ matrix functions, $\Lambda^{(j)}$ are $2Mn_0 \times 2Mn_0$ diagonal matrices.

Discrete chain:

$$W_{t_n} + W_{z_n}W = 0 \quad \Rightarrow \quad W^{(i)}_{t_n} + W^{(1)}_{z_n}W^{(i)} + W^{(i+1)} = 0,$$
$$i = 1, \ldots, N_0, \quad W^{(N_0+1)} = 0, \quad n = 1, 2.$$

Let $i = 1$ and eliminate $W^{(2)}$ to obtain self-dual Yang-Mills equation:

$$W^{(1)}_{z_1 t_2} - W^{(1)}_{z_2 t_1} + W^{(1)}_{z_2}W^{(1)}_{z_1} - W^{(1)}_{z_1}W^{(1)}_{z_2} = 0,$$

2(c) $GL(N)$ SDYM and differential reduction

Let

$$\chi^{(j)} = \begin{pmatrix} \Psi^{(2j-1)} & \Psi^{(2j)} \\ \Psi_x^{(2j-1)} & \Psi_x^{(2j)} \end{pmatrix}, \quad \Lambda^{(j)} = \mathrm{diag}(\tilde{\Lambda}^{(2j-1)}, \tilde{\Lambda}^{(2j)}), \quad j = 1, \ldots, N_0,$$

where $\Psi^{(m)}$ are $Mn_0 \times Mn_0$ matrix functions:

$$\Psi_{xx}^{(m)} = a\Psi^{(m)}\tilde{\Lambda}^{(m)} + \nu\Psi_x^{(m)} + \mu\Psi^{(m)}, \quad m = 1, \ldots, 2N_0,$$

Then the linear eq. for Ψ yields

$$\Psi^{(j)}\Lambda^{(j)} = \sum_{i=0}^{N_0} W^{(i)}\Psi^{(j)}(\Lambda^{(j)})^{-i}, \quad j = 0, 1, \ldots, N_0,$$

where a, ν and μ are $n_0 M \times n_0 M$ diagonal constant matrix parameters and

$$W^{(i)} = \begin{pmatrix} w^{(i)} & v^{(i)} \\ w_x^{(i)} + v^{(i)}\mu + v^{(i+1)}a + v^{(0)}aw^{(i)} & v_x^{(i)} + v^{(i)}\nu + w^{(i)} + v^{(0)}av^{(i)} \end{pmatrix}.$$

The first block-row reads:

$$\Psi\Lambda = \sum_{i=1}^{N_0} \left(v^{(i)} \Psi_x + w^{(i)} \Psi \right) \Lambda^{-i+1}.$$

Ψ is a solution of the system

$$\begin{aligned} & \Psi_{t_n} + \Psi_{z_n} \Lambda = 0, \\ \mathcal{E}^{(0)} \ := \ & \Psi_{xx} = a\Psi\Lambda + \nu\Psi_x + \mu\Psi, \end{aligned}$$

where

$$\begin{aligned} \Psi &= (\Psi^{(1)}, \ldots, \Psi^{(2N_0)}), \\ \Lambda &= \mathrm{diag}(\tilde{\Lambda}^{(1)}, \ldots, \tilde{\Lambda}^{(2N_0)}), \end{aligned}$$

$v^{(i)}$, $w^{(i)}$, $\Psi^{(i)}$ and $\tilde{\Lambda}^{(j)}$ are $Mn_0 \times Mn_0$ matrix functions, a, ν and μ are $Mn_0 \times Mn_0$ diagonal constant matrices.

Compatibility conditions:

$$\Lambda_{t_n} + \Lambda_{z_n}\Lambda = 0, \quad \Lambda_x = 0$$

Evolution part of the nonlinear system

$$\begin{aligned}
E_n^{(1i)} &:= v^{(i)}_{t_n} + v^{(1)}_{z_n}(w^{(i)} + v^{(i)}_x + v^{(i)}\nu + v^{(1)} a v^{(i)}) + w^{(1)}_{z_n} v^{(i)} + v^{(i+1)}_{z_n} = 0, \\
& \quad v^{(N_0+1)} = 0, \quad i = 1, \dots, N_0, \\
E_n^{(0i)} &:= w^{(i)}_{t_n} + v^{(1)}_{z_n}(w^{(i)}_x + v^{(i)}\mu + v^{(1)} a w^{(i)} + v^{(i+1)} a) + w^{(1)}_{z_n} w^{(i)} + w^{(i+1)}_{z_n} = 0, \\
& \quad w^{(N_0+1)} = 0, \quad i = 1, \dots, N_0,
\end{aligned}$$

Non-evolution part of the nonlinear system:

$$\begin{aligned}
\tilde{E}^{(1i)} &:= v^{(i)}_{xx} + [w^{(i)}, \nu] + 2w^{(i)}_x + 2v^{(i)}_x \nu - \nu v^{(i)}_x + [v^{(i)}, \mu] + [v^{(i)}, \nu]\nu + [v^{(1)}, \nu] a v^{(i)} + \\
& \quad 2v^{(1)}_x a v^{(i)} + [w^{(1)}, a] v^{(i)} + [v^{(1)}, a](w^{(i)} + v^{(i)}_x + v^{(i)}\nu + v^{(1)} a v^{(i)}) + [v^{(i+1)}, a] = 0,
\end{aligned}$$

$$\begin{aligned}
\tilde{E}^{(0i)} &:= w^{(i)}_{xx} + [w^{(i)}, \mu] + [w^{(1)}, a] w^{(i)} + 2v^{(i)}_x \mu + [v^{(i)}, \nu]\mu + [v^{(1)}, \nu] a w^{(i)} + \\
& \quad 2v^{(1)}_x a w^{(i)} - \nu w^{(i)}_x + [v^{(1)}, a](w^{(i)}_x + v^{(i)}\mu + v^{(1)} a w^{(i)} + v^{(i+1)} a) + [w^{(i+1)}, a] + \\
& \quad 2v^{(i+1)}_x a + [v^{(i+1)}, \nu] a = 0
\end{aligned}$$

The complete system of PDEs

for $v^{(i)}$ and $w^{(i)}$, $i = 1, 2$ corresponds to $i = 0$:

$$E_n^{(11)} := v_{t_n}^{(1)} + v_{z_n}^{(1)}(w^{(1)} + v_x^{(1)} + v^{(1)}\nu + v^{(1)}av^{(1)}) + w_{z_n}^{(1)}v^{(1)} + v_{z_n}^{(2)} = 0,$$

$$E_n^{(01)} := w_{t_n}^{(1)} + v_{z_n}^{(1)}(w_x^{(1)} + v^{(1)}\mu + v^{(1)}aw^{(1)} + v^{(2)}a) + w_{z_n}^{(1)}w^{(1)} + w_{z_n}^{(2)} = 0,$$

$$\begin{aligned}\tilde{E}_n^{(11)} := \; & v_{xx}^{(1)} + [w^{(1)}, \nu] + 2w_x^{(1)} + 2v_x^{(1)}\nu - \nu v_x^{(1)} + [v^{(1)}, \mu] + \\ & [v^{(1)}, \nu](\nu + av^{(1)}) + 2v_x^{(1)}av^{(1)} + [w^{(1)}, a]v^{(1)} + \\ & [v^{(1)}, a](w^{(1)} + v_x^{(1)} + v^{(1)}\nu + v^{(1)}av^{(1)}) + [v^{(2)}, a] = 0,\end{aligned}$$

$$\begin{aligned}\tilde{E}^{(01)} := \; & w_{xx}^{(1)} + [w^{(1)}, \mu] + [w^{(1)}, a]w^{(1)} + 2v_x^{(1)}\mu + [v^{(1)}, \nu]\mu + [v^{(1)}, \nu]aw^{(1)} + \\ & 2v_x^{(1)}aw^{(1)} - \nu w_x^{(1)} + [v^{(1)}, a](w_x^{(1)} + v^{(1)}\mu + v^{(1)}aw^{(1)} + v^{(2)}a) + [w^{(2)}, a] + \\ & 2v_x^{(2)}a + [v^{(2)}, \nu]a = 0\end{aligned}$$

2(d). Frobenius reduction and associated higher dimensional systems of nonlinear PDEs

$$v^{(i)} = \{v^{(i;kl)},\ \ k,l = 1,\dots,n_0\},\ \ w^{(i)} = \{w^{(i;kl)},\ \ k,l = 1,\dots,n_0\},$$
$$v^{(i;kl)} = \delta_{k1} v^{(i;l)} + \delta_{k(l+n_1(i))} I_M,$$
$$w^{(i;kl)} = \delta_{k1} w^{(i;l)} + \delta_{k(l+n_2(i))} I_M,$$

where n_0 is an integer parameter, $v^{(i;l)}$ and $w^{(i;l)}$ are $M \times M$ matrix fields, ν, μ and a have the following diagonal forms:

$$\nu = \mathrm{diag}(\underbrace{\tilde\nu,\dots,\tilde\nu}_{n_0}),\ \ \mu = \mathrm{diag}(\underbrace{\tilde\mu,\dots,\tilde\mu}_{n_0}),\ \ a = \mathrm{diag}(\underbrace{\tilde a,\dots,\tilde a}_{n_0}),$$

where $\tilde\nu$, $\tilde\mu$ and $\tilde a$ are $M \times M$ diagonal matrices.

Block structure of the nonlinear PDEs:

$$E_n^{(mi)} = \{E_n^{(mi;l)} \delta_{k1},\ \ k,l, = 1,\dots,n_0\},$$
$$\tilde E_n^{(mi)} = \{\tilde E_n^{(mi;l)} \delta_{k1},\ \ k,l, = 1,\dots,n_0\},\ \ m = 0,1,$$

Explicit form of the nonlinear PDEs, $n_1(i) = n_2(i) = 1$

$$E_n^{(11;1)} := v_{t_n}^{(1;1)} + v_{z_n}^{(1;1)}(w^{(1;1)} + v_x^{(1;1)} + v^{(1;1)}\nu) + (v_{z_n}^{(1;1)}v^{(1;1)} + v_{z_n}^{(1;1+n_1(0))})\tilde{a}v^{(1;1)} + v_{z_n}^{(1;1)}v^{(1;1+n_1(0))}\tilde{a} + w_{z_n}^{(1;1)}v^{(1;1)} + (Q_1^{(1;1)})_{z_n} = 0,$$

$$E_n^{(01;1)} := w_{t_n}^{(1;1)} + v_{z_n}^{(1;1)}(w_x^{(1;1)} + v^{(1;1)}\mu + v^{(2;1)}\tilde{a}) + (v_{z_n}^{(1;1)}v^{(1;1)} + v_{z_n}^{(1;1+n_1(0))})\tilde{a}w^{(1;1)} + v_{z_n}^{(1;1)}v^{(1;1+n_2(0))}\tilde{a} + w_{z_n}^{(1;1)}w^{(1;1)} + (Q_2^{(1;1)})_{z_n} = 0,$$

$$\tilde{E}^{(11;l)} := \tilde{F}^{(11;l)}(v^{(1;1)}, w^{(1;1)}, v^{(1;2)}) + [Q_1^{(1;1)}, \tilde{a}],$$

$$\tilde{E}^{(01;l)} := \tilde{F}^{(11;l)}(v^{(1;1)}, w^{(1;1)}, v^{(2;1)}, v^{(1;2)} + [Q_2^{(1;1)}, \tilde{a}]$$

where

$$Q_1^{(1;1)} = v^{(1;3)}\tilde{a} + v^{(1;2)} + v^{(1;2)}\tilde{\nu} + w^{(1;2)} + v^{(2;1)},$$

$$Q_2^{(1;l)} = v^{(1;3)}\tilde{a} + v^{(1;2)}\tilde{\mu} + v^{(1;2)}\tilde{a} + w^{(1;2)} + w^{(2;1)}.$$

The complete system of PDEs
for the matrix fields $v^{(1;1)}$, $w^{(2;1)}$, $v^{(2;1)}$, $v^{(1;2)}$:

$$(E_1^{(11;1)})_{z_2} - (E_2^{(11;1)})_{z_1} = 0,$$

$$(E_1^{(01;1)})_{z_2} - (E_2^{(01;1)})_{z_1} = 0,$$

$$[E_1^{(11;1)}, \tilde{a}] - (\tilde{E}^{(11;1)})_{z_1} = 0,$$

$$[E_1^{(01;1)}, \tilde{a}] - (\tilde{E}^{(01;1)})_{z_1} = 0.$$

The scalar case: $u \equiv v^{(1;1)}$

$$(E_1^{(11;1)})_{z_2} - (E_2^{(11;1)})_{z_1} = 0 \quad \Leftrightarrow \quad u_{z_1 t_2} - u_{z_2 t_1} + u_{z_2} u_{z_1 x} - u_{z_1} u_{z_2 x} = 0.$$

2(e). Solutions

Starting equation, $M = 1$:

$$\chi\Lambda = W\chi$$

After all reductions, this matrix equation may be splitted into two subsystems of scalar equations:

First subsystem:,

$$\begin{aligned} \Psi^{(l;1m)}\tilde{\Lambda}^{(l;m)} &= \sum_{i=1}^{N_0}\sum_{j=1}^{n_0}\left[v^{(i;j)}\Psi_x^{(l;jm)} + \; w^{(i;j)}\Psi^{(l;jm)}\right](\tilde{\Lambda}^{(l;m)})^{-i+1}, \\ & l = 1,\ldots,2N_0 \;\;,\;\; m = 1,\ldots,n_0. \end{aligned}$$

This is the system of $2N_0n_0$ linear algebraic equations for the same number of matrix fields $v^{(i;l)}$ and $w^{(i;l)}$, $i = 1,\ldots,N_0$, $j = 1,\ldots,n_0$.

Second subsystem: $n > 1$,

$$\begin{aligned} \Psi^{(l;nm)}\tilde{\Lambda}^{(l;m)} &= \sum_{i=0}^{N_0}\left[\Psi_x^{(l;(n-n_1(i))m)} + \Psi^{(l;(n-n_2(i))m)}\right](\tilde{\Lambda}^{(l;m)})^{-i+1}, \\ \Psi^{(l;ij)} &= 0, \text{ if, } i \le 0, \\ & l = 1,\ldots,2N_0 \;\;,\;\; n,m = 1,\ldots,n_0, \end{aligned}$$

This equation expresses recursively the functions $\Psi^{(l;nm)}$, $n > 1$, in terms of the functions $\Psi^{(l;1m)}$ and their x-derivatives.

Functions $\Psi^{(l;nm)}$ and $\tilde\Lambda^{(l;m)}$ are defined as follows:

$$\Psi^{(l;1m)}(\vec{x}) = \sum_{i=1}^{2} \psi^{(lm;i)}(z_1 - \tilde\Lambda^{(l;m)} t_1, z_2 - \tilde\Lambda^{(l;m)} t_2) e^{k^{(lm;i)}x}$$

$$\tilde\Lambda^{(l;m)} = E^{(lm)}(z_1 - \tilde\Lambda^{(l;m)} t_1, z_2 - \tilde\Lambda^{(l;m)} t_2),$$

$$k^{(lm;1)} = \frac{1}{2}\left(\tilde\nu + \sqrt{\tilde\nu^2 + 4\tilde a \tilde\Lambda^{(l;m)} + 4\tilde\mu}\right), \quad k^{(lm;1)} = \frac{1}{2}\left(\tilde\nu - \sqrt{\tilde\nu^2 + 4\tilde a \tilde\Lambda^{(l;m)} + 4\tilde\mu}\right),$$

$$m = 1, \dots, n_0, \quad l = 1, \dots, 2N_0.$$

Simplest example of the solution: $N_0 = n_0 = 1$

$$\Psi^{(l;11)}(\vec{x}) = \sum_{i=1}^{2} \psi^{(l1;i)}(z_1 - \tilde{\Lambda}^{(l;1)}t_1, z_2 - \tilde{\Lambda}^{(l;1)}t_2)e^{k^{(l1;i)}x}, \quad l = 1, 2,$$

$$\tilde{\Lambda}^{(l;1)} = E^{(l1)}(z_1 - \tilde{\Lambda}^{(l;1)}t_1, z_2 - \tilde{\Lambda}^{(l;1)}t_2), \quad l = 1, 2.$$

Then

$$u \equiv v^{(1;1)} = \frac{\Delta_1}{\Delta},$$

$$\Delta = \sum_{i,j=1}^{2} \Big((-1)^{i+1}K_1 - (-1)^{j+1}K_2\Big) e^{\left(\tilde{\nu}+(-1)^{i+1}K_1+(-1)^{j+1}K_2\right)x} \psi^{(11;i)}(y_1^1, y_2^1)\psi^{(21;j)}(y_1^2, y_2^2),$$

$$K_l = \frac{1}{2}\sqrt{\tilde{\nu}^2 + 4\tilde{a}\tilde{\Lambda}^{(l;1)} + 4\tilde{\mu}}, \quad y_1^l = z_1 - \tilde{\Lambda}^{(l;1)}t_1, \quad y_2^l = z_2 - \tilde{\Lambda}^{(l;1)}t_2,$$

$$\Delta_1 = (\tilde{\Lambda}^{(1;1)} - \tilde{\Lambda}^{(2;1)}) \sum_{i,j=1}^{2} e^{\left(\tilde{\nu}+(-1)^{i+1}K_1+(-1)^{j+1}K_2\right)x} \psi^{(11;i)}(y_1^1, y_2^1)\psi^{(21;j)}(y_1^2, y_2^2)$$

Solution u has no singularities if $\Delta \neq 0$: $K_1 > K_2 > 0$, $\psi^{(11;2)} < 0$, $\psi^{(11;1)}, \psi^{(21;1)}, \psi^{(21;2)} > 0$.
Since $\tilde{\Lambda}^{(l;1)}$ is implicitly given by the eq.(1), constructed function u describes the break of the wave profile unless $\tilde{\Lambda}^{(l;1)} = const$.

For instance:

$$K_2 = 0 \quad \Rightarrow \quad \tilde{\Lambda}^{(2;1)} = -\frac{\tilde{\nu}^2 + 4\tilde{\mu}}{4\tilde{a}} = const,$$

$$\psi^{(11;1)}(y_1^1, y_2^1) = \xi_1(y_1^1, y_2^1) > 0, \quad \psi^{(11;2)}(y_1^2, y_2^2) = -\xi_2(y_1^2, y_2^2) < 0.$$

One has

$$u = \frac{(\tilde{\Lambda}^{(1;1)} - \tilde{\Lambda}^{(2;1)})\left(e^{K_1 x}\xi_1(y_1^1, y_2^1) - e^{-K_1 x}\xi_2(y_1^2, y_2^2)\right)}{K_1\left(e^{K_1 x}\xi_1(y_1^1, y_2^1) + e^{-K_1 x}\xi_2(y_1^2, y_2^2)\right)},$$

$$y_1^l = z_1 - \tilde{\Lambda}^{(l;1)} t_1, \quad y_2^l = z_2 - \tilde{\Lambda}^{(l;1)} t_2,$$