Stability and collapse of solitons near transition from supercritical to subcritical bifurcation in hydrodynamical systems.

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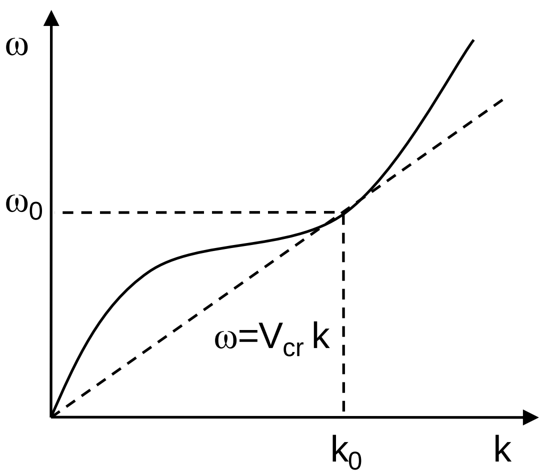
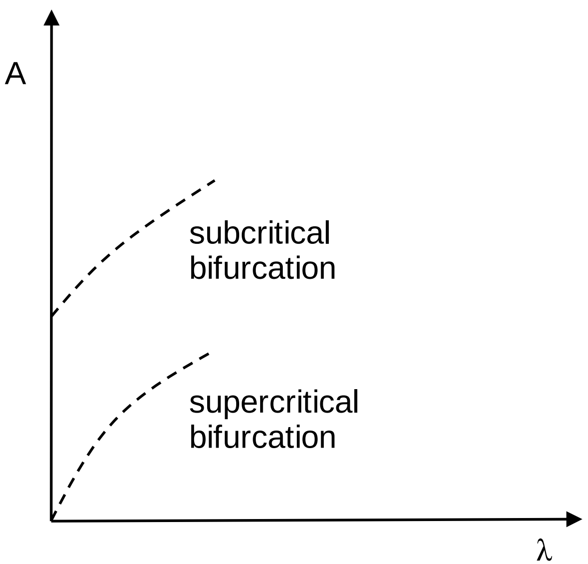
According to usual definition, solitons are localized objects moving with some constant velocity **V**. Soliton velocity may take values inside some interval of allowed velocity values. Borders of this interval are defined from the condition of touching of the plane **ω=kV** and the dispersion surface **ω=ω(k)**. The latter means that the borders of soliton velocity interval are the points there the phase velocity coincides with the group velocity.



As $V \to V_{cr}$, soliton amplitude may turn to zero smoothly (supercritical bifurcation), or where can be a step (subcritical bifurcation). Supercritical bifurcation is analogous to phase transitions of second order, while subcritical bifurcation is analogous to phase transitions of first order.

It turns out, that in case of supercritical bifurcation solitons have universal behavior near the point of bifurcation: their shape is defined by the stationary focusing NLSE,

$$\Delta\psi - \lambda^2\psi - \mu|\psi|^2\psi = 0, \quad \lambda^2 = (V_{cr} - V)/V_{cr}, \quad \mu = Sign(\omega'' \tilde{T}_{k_0k_0k_0k_0}),$$



their amplitude is proportional to $|V_{cr}-V|^{1/2}$, and their width - - to $|V_{cr}-V|^{-1/2}$.

The question here arises: what can we say about solitons behavior in the opposite case of subcritical bifurcation, at least near the transition point from supercritical bifurcation to subcritical bifurcation?

Effective interaction Hamiltonian for the envelope of the first harmonic of soliton is:

$$H_{\mathrm{int}} = \frac{1}{2}\int(\widehat{T}_{k_0k_0k_0k_0} + \Delta T)a_1 * a_2 * a_3 a_4 \delta_{1+2-3-4} dk_{1234} +$$

$$+\frac{1}{3}\int \widehat{C}_{k_0k_0k_0k_0k_0k_0} a_1 * a_2 * a_3 * a_4 a_5 a_6 \delta_{1+2+3-4-5-6} dk_{123456}$$

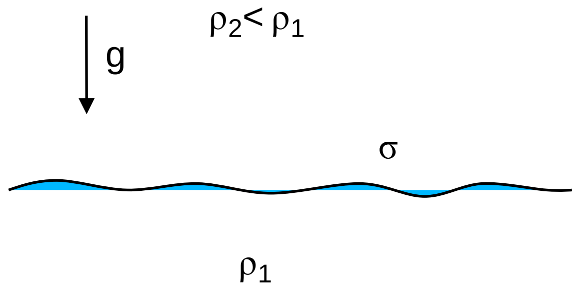
Equation of motion for the envelope of the first harmonic:

$$i\psi_t - \lambda^2\psi + \frac{\omega_0''}{2}\psi_{xx} - \mu|\psi|^2\psi + 4i\beta|\psi|^2\psi_x + \gamma\psi\hat{k}|\psi|^2 + 3C|\psi|^4\psi = 0$$

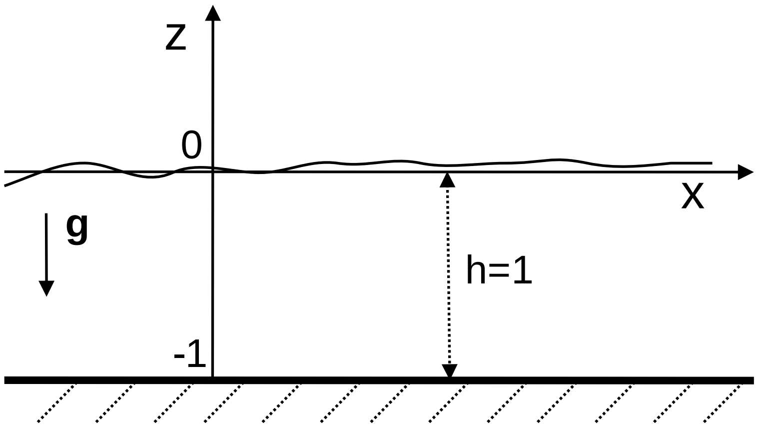
where $k = -\partial_x H$, $\widehat{H}f(x) = P.V.\int_{-\infty}^{+\infty}\frac{f(x')dx'}{x'-x}$ is the Hilbert transform.

Examples of such systems are:

1. Internal wave on the interface of division between 2 ideal incompressible liquids in gravity field and with presence of surface tension. Fluids are supposed to be potential. Minimal phase velocity exists in the system, type of bifurcation depends only on the density ratio of the liquids **ε**: the bifurcation is supercritical if **$\varepsilon<\varepsilon_{cr}$ ≈**0.28 and subcritical vice versa (these two facts were first shown by F.Dias & G.Iooss, 1996)..



2. Ideal incompressible liquid with free surface in gravity field, fluid is supposed to be potential.



It is known, that in such a system a solution in the form of a periodic wave with wave vector **k** is unstable with respect to small modulations if **kh=k>k$_{cr}$≈1.363** (Benjamin-Feir instability). In the opposite case of **k<k$_{cr}$** such solution is stable. From the other hand, quasimonochromatic solutions in the form of envelope solitons exist and stable in case of **k>k$_{cr}$**, and do not exist in the opposite case. So, the question here arises about the behavior of the system when the wave vector is close to it's critical value.

$$i\psi_t - \lambda^2\psi + \frac{\omega_0''}{2}\psi_{xx} - \mu|\psi|^2\psi + 4i\beta|\psi|^2\psi_x + \gamma\psi\hat{k}|\psi|^2 + 3C|\psi|^4\psi = 0$$

1. Internal wave system:

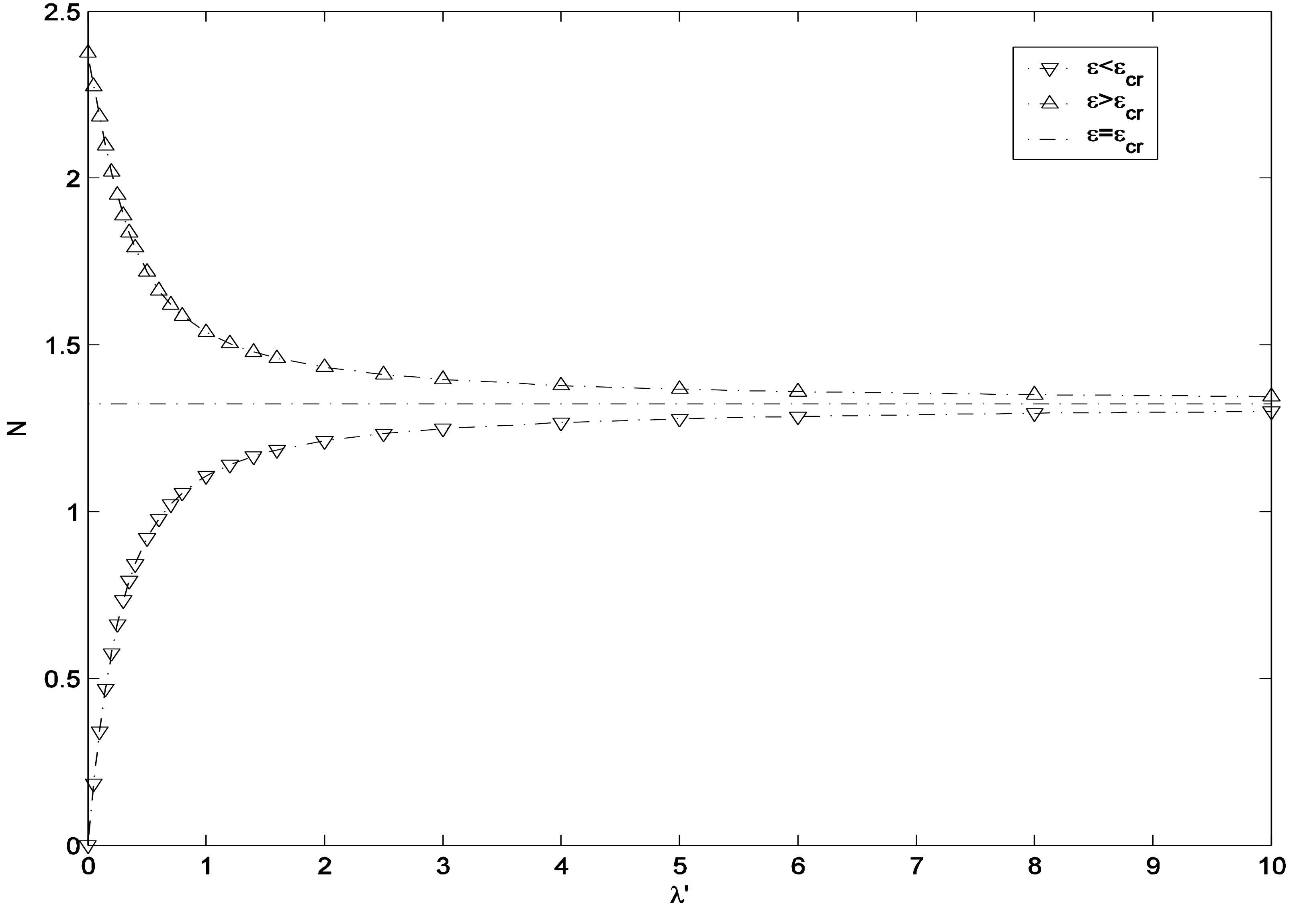
$$\mu = \mathrm{sign}(\varepsilon - \varepsilon_{cr}), \quad \beta = 6/\sqrt{427}, \quad \gamma = 32/\sqrt{427}, \quad \mathrm{C}=319/1281,$$

$$\lambda \sim (V_{cr} - V)^{1/2}|\rho - \rho_{cr}|^{-1}.$$

2. Liquid with finite depth:

$$\mu = \mathrm{sign}(k_{cr} - k), \quad \beta \approx -0.397, \quad \gamma = 0, \quad C \approx 0.176.$$

Dependence of wave action **N** on parameter **λ**:



Solitons here are stationary points of energy **E** with fixed wave action **N**:

$$\delta(E+\lambda^2 N)=0,$$

where **E** and **N** can be written in dimensionless variables as:

$$E=\int\left[\,|\psi_x|^2+\frac{\mu}{2}|\psi|^4+i\beta(\psi_x^*\psi-\psi_x\psi^*)|\psi|^2-\frac{\gamma}{2}|\psi|^2\,\hat{k}\,|\psi|^2-C|\psi|^6\right]dx,$$

$$N=\int|\psi|^2 dx.$$

We use Lyapunov theorem for stability analysis of soliton solutions. It turns out that in case of focusing cubic nonlinearity **μ<0** solitons realize minimum of energy **E** with fixed wave action **N**. In case of defocusing nonlinearity **μ>0** solitons are saddle points for energy, that should mean that they are unstable at least with respect to finite perturbations.

Here we investigate nonlinear stage of instability of solitons, corresponding to case of subcritical bifurcation, near the transition between supercritical bifurcation to subcritical one, in systems described by equations of the type:

$$i\psi_t + \psi_{xx} - \lambda^2\psi - \mu|\psi|^2\psi + 4i\beta|\psi|^2\psi_x + 3C|\psi|^4\psi + \gamma\psi(\widehat{k}|\psi|^2) = 0.$$

In case of zeroth self-steepening (**β=0**), it is possible to write down the collapse criteria:

$$R_{tt} = 8\left(E - \frac{\mu}{4}\int|\psi|^4\,dx\right) < 8E \qquad \rightarrow \qquad R \leq 4Et^2 + \alpha_1 t + \alpha_2$$

for positive-defined value **R**:

$$R = \int x^2\,|\psi|^2\,dx.$$

Form the other hand, in case of all terms taken into account, it was shown that the amplitude of a distribution with negative energy **E** cannot be less than some fixed value defined from a combination of integrals of motion:

$$\max|\psi|^4 \geq -\frac{3E}{N}$$

At last, dynamics of perturbed soliton solutions corresponding to subcritical bifurcation was investigated numerically with the help of Runge-Kutta schema of the 4th order.

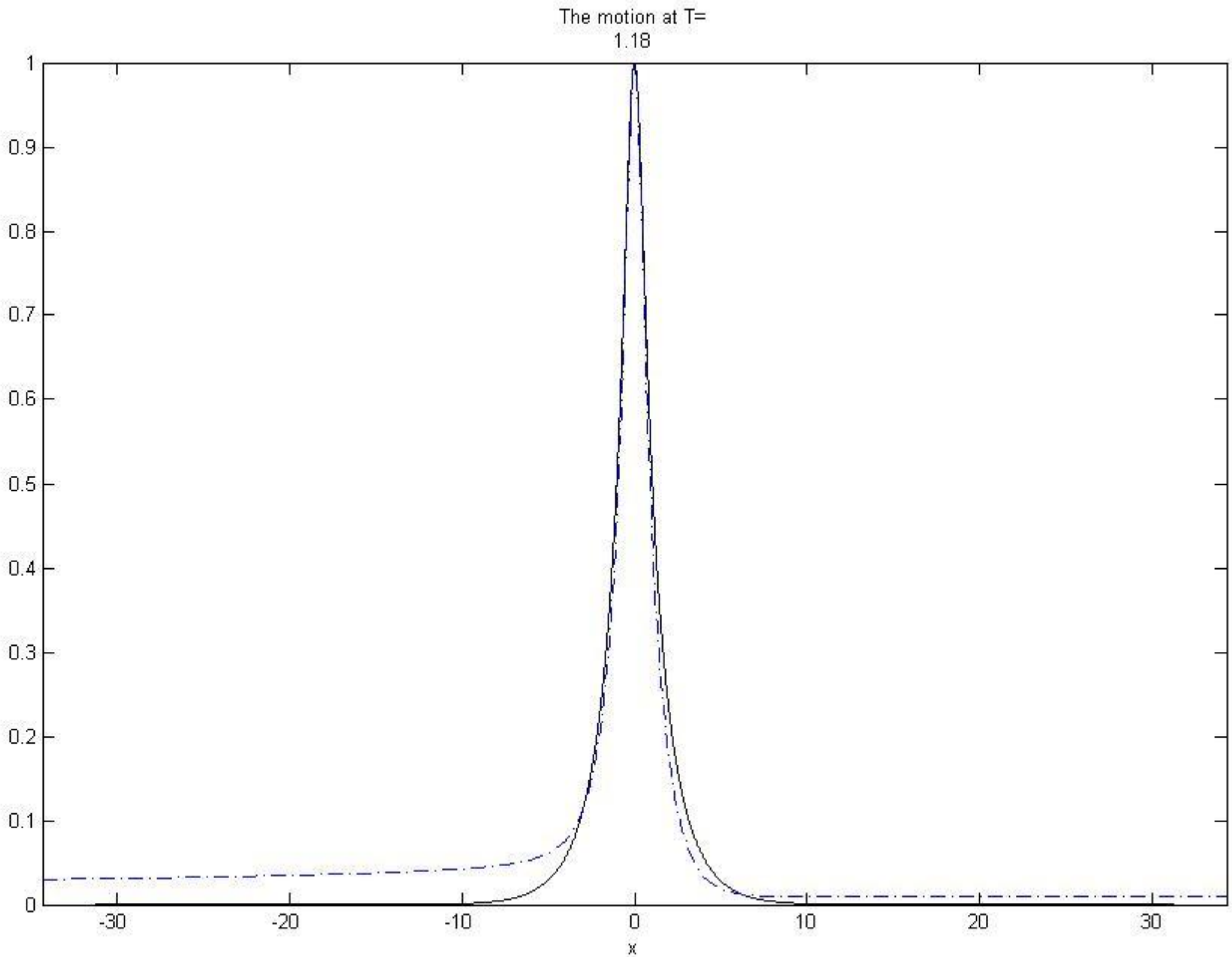
Initial perturbation of energy consisted in increasing of soliton's amplitude by 0.1%-10% led to self-similar collapse of such distributions:

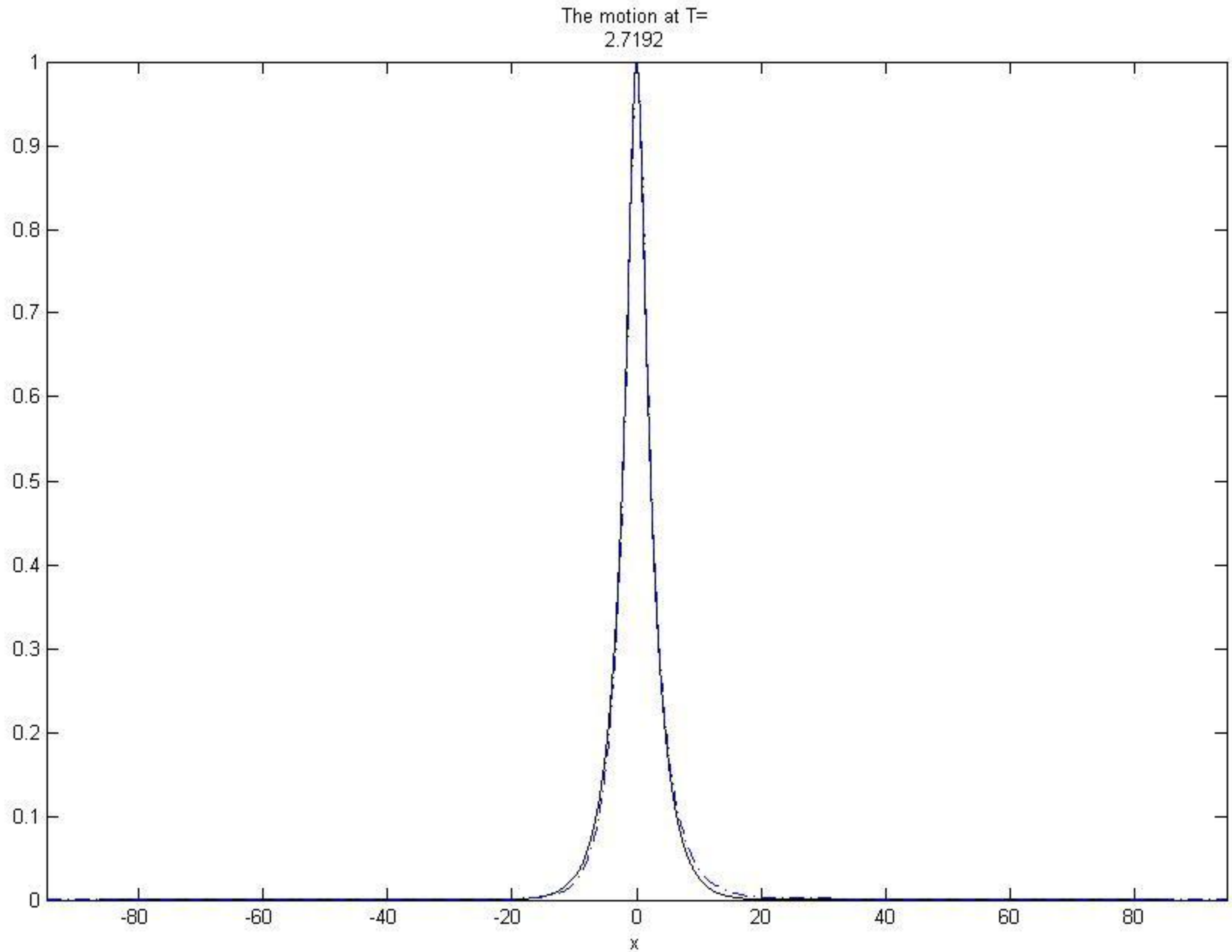
$$r(x,t) = (t_0 - t)^{-1/4} f\left(\frac{x}{(t_0 - t)^{1/2}}\right),$$

where $\mathbf{t_0}$ is the collapse time.

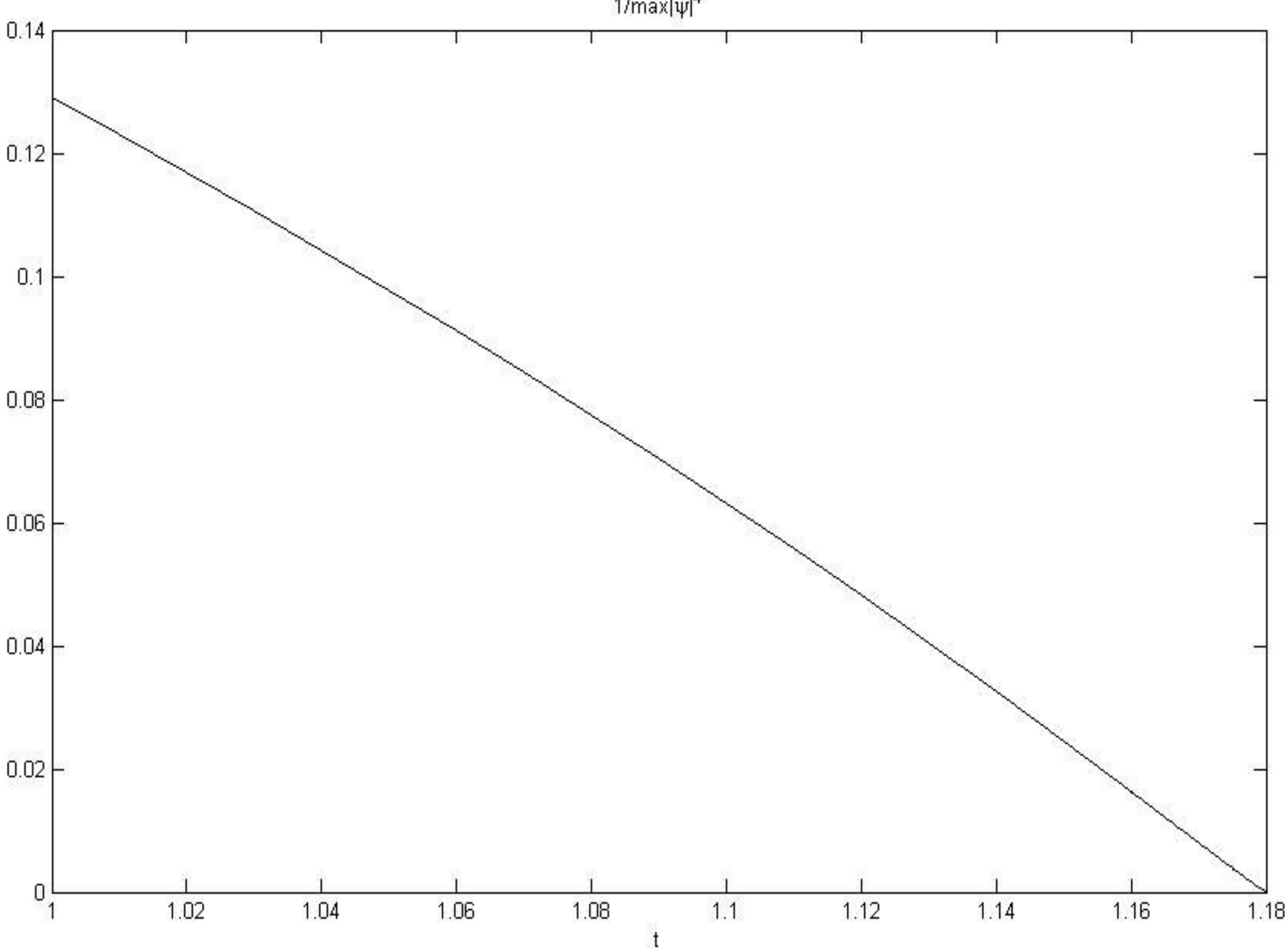
From the other hand, decreasing of amplitude by the same values led to diffusing of the initial pulse. Such situation is quite usual, in the sense that solitons are often talked about as objects where nonlinearity is compensated with dispersion.

Results of calculations are quite interesting because the presence of self-steepening $\mathbf{i|\psi|^2\psi_x}$ should lead to breaking of the wave. Nevertheless, we do not have any breaking: peaks are almost symmetric while all nonsymmetry arise on the tails of distributions.


The motion at T=
1.18
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-30
-20
-10
0
10
20
30
x



The motion at T=
2.7192
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-80
-60
-40
-20
0
20
40
60
80
x



1/max|ψ|^4
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1
1.02
1.04
1.06
1.08
1.1
1.12
1.14
1.16
1.18
t

Papers:

- D.S. Agafontsev, F. Dias, E.A. Kuznetsov, *Collapse of solitary waves near transition from supercritical to subcritical bifurcations*, arXiv:0805.1620v1, JETP Letters vol. 87 (2008), iss. 11, p.767.
- D.S.Agafontsev, F.Dias, E.A.Kuznetsov, *Bifurcations of Solitary Waves*, Journal of Mathematical Physics, Analysis and Geometry, vol. 4 (2008), iss. 4, pp. 529-550.